An overview of methods for deriving the radiative transfer theory from the Maxwell equations. II: Approach based on the Dyson and Bethe–Salpeter equations

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Abstract

In this paper, the vector radiative transfer equation is derived by means of the vector integral Foldy equations describing the electromagnetic scattering by a group of particles. By assuming that in a discrete random medium the positions of the particles are statistically independent and by applying the Twersky approximation to the order-of-scattering expansion of the total field, we derive the Dyson equation for the coherent field and the ladder approximated Bethe–Salpeter equation for the dyadic correlation function. Then, under the far-field assumption for sparsely distributed particles, the Dyson equation is reduced to the Foldy integral equation for the coherent field, while the iterated solution of the Bethe–Salpeter equation ultimately yields the vector radiative transfer equation.

Keywords:
Electromagnetic scattering
Frequency-domain macroscopic electromagnetics
Discrete random media
Dyson equation
Bethe–Salpeter equation
Radiative transfer theory

1 Introduction

In the first part of this series [1], we derived the vector radiative transfer equation for a discrete random layer with non-scattering boundaries by invoking at the very outset the algebraic far-field approximation to the Foldy equations [2, 3, 4] applicable to sparsely distributed particles. In other words, we assumed from the very beginning that each particle is located in the far zones of all the other particles and that the observation point is also located in the far zone of any particle. The coherent field and the vector radiative transfer equation were obtained by applying the Twersky approximation [5] to the far-field order-of-scattering expansion of the total field and by taking the configuration average of the resulting equations under the assumption that the positions of the particles are statistically independent.

In this paper we analyze how the use of the far-field assumption in the derivation of the radiative transfer equation can we delayed as far as possible. We therefore use the exact integral Foldy equations of electromagnetic scattering [6, 7, 8] formulated in terms of the transition dyadic of an individual particle. For a discrete random medium with uncorrelated particle positions, these are employed in conjunction with the Twersky approximation to derive the Dyson equation for the coherent field and the ladder-approximated Bethe–Salpeter equation for the dyadic correlation function. The coherent field and the vector radiative transfer equation are then obtained by simplifying the Dyson and Bethe–Salpeter equations via the far-field assumption. The final section discusses the similarities of and differences between the two approaches to arrive at the same radiative transfer equation.

We have tried to make this paper maximally self-contained while keeping its size manageable. To this end, we assume that the reader is already familiar with Ref. [1] and use the same conceptual base and notation. Neither Ref. [1] nor this second part are intended for a complete novice in the field of electromagnetic scattering; for the basics, we refer to the tutorial [8] and introductory text [9].

2 Dyson and Bethe–Salpeter equations

We consider the same scattering geometry as in Ref. [1]. More precisely, a group of N identical, homogeneous, nonmagnetic particles with permittivity ε_2 are placed in a lossless, homogeneous, nonmagnetic, and isotropic medium with permittivity ε_1 and permeability μ_0. The wavenumbers in the background medium and the particle are $k_1 = \omega\sqrt{\varepsilon_1\mu_0}$ and $k_2 = \omega\sqrt{\varepsilon_2\mu_0}$, respectively, where ω is the angular frequency. The particles are centered at $\mathbf{R}_1$, $\mathbf{R}_2$, ..., $\mathbf{R}_N$, the origins of the particles are confined to a macroscopically plane-parallel layer with non-scattering plane boundaries. The domain occupied by particle i is denoted by D_i, the domain populated by the particles is denoted by D, and the particulate medium is characterized by the particle number concentration n_0. The particles have the same orientation, and the coordinate systems of the particles are aligned with the global coordinate system. The incident field is

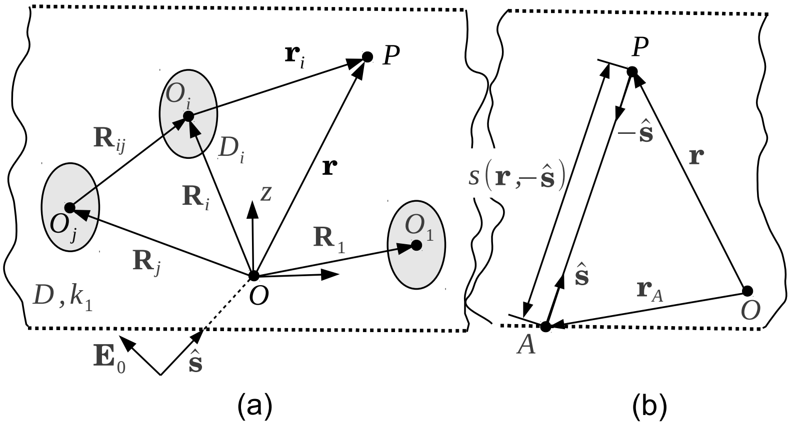

Figure 1: (a) A group of particles confined to a macroscopically plane-parallel layer with non-scattering boundaries, and (b) the distance $s(\mathbf{r}, -\widehat{\mathbf{s}}) = \widehat{\mathbf{s}} \cdot (\mathbf{r} - \mathbf{r}_A)$, where $\mathbf{r}_A$ is the point where the straight line with the direction vector $-\widehat{\mathbf{s}}$ going through the observation point P crosses the lower boundary of the layer.

a plane electromagnetic wave with a wavenumber k_1, propagation direction $\widehat{\mathbf{s}}$, and amplitude $\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}})$, i.e.,

$$\mathbf{E}_0(\mathbf{r}) = \boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) e^{\mathrm{j} k_1 \widehat{\mathbf{s}} \cdot \mathbf{r}}, \tag{1}$$

with $\mathrm{j} = \sqrt{-1}$.

Throughout this paper we will use integral-operator notation to write integral equations in a compact form [6]. This short-hand notation is introduced in Appendix 1 along with the Fourier transforms of vector and dyadic functions. Moreover, the transition dyadic of an individual particle, which is the key ingredient of our analysis, is defined in Appendix 2 (see also the recent tutorial [8]).

The frequency-domain scattering by a group of fixed particles can be described by the vector Foldy equations [2, 3, 7, 8] which can be formulated with respect to either the electric fields or the dyadic Green's functions. For the electric fields, the Foldy equations read

$$\mathbf{E} = \mathbf{E}_0 + \sum_i \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \mathbf{E}_{\mathrm{exc}i}, \tag{2}$$

$$\mathbf{E}_{\mathrm{exc}i} = \mathbf{E}_0 + \sum_{j \neq i} \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_j \, \mathbf{E}_{\mathrm{exc}j}, \tag{3}$$

where $\mathbf{E}$ is the total field, $\mathbf{E}_0$ is the incident field, $\mathbf{E}_{\mathrm{exc}i}$ is the field "exciting" particle i, $\overline{\mathbf{T}}_i$ is the transition dyadic of particle i, $\overline{\mathbf{G}}_0$ is the dyadic Green's function in free space, and the summations run implicitly from 1 to N. The transition dyadic can be thought of as being a unique scattering ID of a particle in that it expresses the field scattered by the particle everywhere in space in terms of the field inside the particle [8]. Eq. (2) gives the total field, while Eq. (3) is an integral equation for the exciting fields. For the dyadic Green's functions, the Foldy equations are [6]

$$\overline{\mathbf{G}} = \overline{\mathbf{G}}_0 + \sum_i \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \overline{\mathbf{G}}_i, \tag{4}$$

$$\overline{\mathbf{G}}_i = \overline{\mathbf{G}}_0 + \sum_{j \neq i} \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_j \, \overline{\mathbf{G}}_j, \tag{5}$$

where $\overline{\mathbf{G}}$ is the dyadic Green's function of the entire group of particles, and $\overline{\mathbf{G}}_i$ is the ith particle Green's dyadic.

The scattering by a discrete random medium is governed by the Dyson equations for the coherent field $\langle \mathbf{E}(\mathbf{r}) \rangle$ and the average dyadic Green's function $\left\langle \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right\rangle$ as well as by the Bethe–Salpeter equation for the dyadic correlation function $\left\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^{\star}(\mathbf{r}') \right\rangle$, where $\otimes$ is the dyadic product sign and the asterisk stands for "complex conjugate". The kernels of these integral equations are the dyadic mass operator $\overline{\mathbf{M}}$ and the tetradic scattering intensity operator $\overline{\overline{\mathbf{I}}}$, respectively [6]. The derivation of these integral equations under the Twersky approximation is given below.

2.1 The Dyson equation for the coherent field

Consider the Foldy equations for the total field (2) and the exciting fields (3). Inserting the iterated solution of the exciting fields equation

$$\begin{aligned} \mathbf{E}_{\mathtt{exc}i} = \mathbf{E}_0 &+ \sum_{j \neq i} \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_j \, \mathbf{E}_0 \\ &+ \sum_{j \neq i} \sum_{k \neq j} \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_j \, \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_k \, \mathbf{E}_0 + \cdots \end{aligned} \tag{6}$$

into the total field equation, and employing the Twersky approximation [5], gives the following order-of-scattering expansion for the total field:

$$\begin{aligned} \mathbf{E} = \mathbf{E}_0 &+ \sum_i \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \mathbf{E}_0 + \sum_i \sum_{j \neq i} \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_j \, \mathbf{E}_0 \\ &+ \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_j \, \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_k \, \mathbf{E}_0 + \cdots . \end{aligned} \tag{7}$$

Recall that the Twersky approximation is based on keeping only the self-avoiding multi-particle sequences and is valid in the limit $N \to \infty$ [1, 5]. Hereafter, we assume that the Neumann series for the total field (7) converges. Denoting

$$
\begin{aligned}
\mathbf{E}_0(\mathbf{r}) &= {}_{\mathbf{r}}\blacktriangleleft\!\!-\!\!- \overset{\text{def}}{=} \blacktriangleleft\!\!-\!\!-,\\
\overline{\mathbf{T}}_i(\mathbf{r},\mathbf{r}') &= {}_{\mathbf{r}}\overset{i}{\circ}_{\mathbf{r}'} \overset{\text{def}}{=} \overset{i}{\circ},\\
\overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}') &= {}_{\mathbf{r}}\!-\!\!-_{\mathbf{r}'} \overset{\text{def}}{=} -\!\!-,
\end{aligned}
$$

we illustrate the order-of-scattering expansion for the total field as

$$
\mathbf{E}(\mathbf{r}) = \blacktriangleleft\!\!-\!\!- + \sum_i -\!\!-\overset{i}{\circ}\blacktriangleleft\!\!-\!\!- + \sum_{i,j;\, j\neq i} -\!\!-\overset{i}{\circ}-\!\!-\overset{j}{\circ}\blacktriangleleft\!\!-\!\!- + \cdots . \tag{8}
$$

Assuming full ergodicity of the N-particle ensemble [9, 10] and taking the conditional configuration average of Eq. (7) under the assumption that the *positions of the particles are uncorrelated*, yields

$$
\begin{aligned}
\langle \mathbf{E} \rangle = \mathbf{E}_0 &+ n_0 \int_D \overline{\mathbf{G}}_0\, \overline{\mathbf{T}}_i\, \mathbf{E}_0\, \mathrm{d}^3\mathbf{R}_i \\
&+ n_0^2 \int_D \overline{\mathbf{G}}_0\, \overline{\mathbf{T}}_i\, \overline{\mathbf{G}}_0\, \overline{\mathbf{T}}_j\, \mathbf{E}_0\, \mathrm{d}^3\mathbf{R}_j \mathrm{d}^3\mathbf{R}_i \\
&+ n_0^3 \int_D \overline{\mathbf{G}}_0\, \overline{\mathbf{T}}_i\, \overline{\mathbf{G}}_0\, \overline{\mathbf{T}}_j\, \overline{\mathbf{G}}_0\, \overline{\mathbf{T}}_k\, \mathbf{E}_0\, \mathrm{d}^3\mathbf{R}_k \mathrm{d}^3\mathbf{R}_j \mathrm{d}^3\mathbf{R}_i + \cdots .
\end{aligned} \tag{9}
$$

In explicit form, the sum of the first two terms in the series (9) is

$$
\begin{aligned}
\mathbf{E}_1(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0 \int_D \Big[\int_{D_i} &\overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1) \\
&\cdot \overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2) \cdot \mathbf{E}_0(\mathbf{r}_2)\, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 \Big]\, \mathrm{d}^3\mathbf{R}_i .
\end{aligned} \tag{10}
$$

Extending the domain of definition of the transition dyadic $\overline{\mathbf{T}}_i$ to the whole D according to

$$
\overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2) = \overline{\mathbf{0}} \text{ for } \mathbf{r}_1 \text{ and/or } \mathbf{r}_2 \notin D_i, \tag{11}
$$

where $\overline{\mathbf{0}}$ is the zero dyad, and interchanging the order of the integrations in Eq. (10) gives

$$
\begin{aligned}
\mathbf{E}_1(\mathbf{r}) &= \mathbf{E}_0(\mathbf{r}) + n_0 \int_D \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1) \\
&\quad \cdot \Big[\int_D \overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2)\, \mathrm{d}^3\mathbf{R}_i \Big] \cdot \mathbf{E}_0(\mathbf{r}_2)\, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 \\
&= \mathbf{E}_0(\mathbf{r}) + n_0 \int_D \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1) \cdot \overline{\mathbf{M}}(\mathbf{r}_1,\mathbf{r}_2) \cdot \mathbf{E}_0(\mathbf{r}_2)\, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2,
\end{aligned} \tag{12}
$$

where the so-called dyadic mass operator is defined by [6]

$$
\overline{\mathbf{M}}(\mathbf{r}_1,\mathbf{r}_2) = n_0 \int_D \overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2)\, \mathrm{d}^3\mathbf{R}_i . \tag{13}
$$

Thus, the dyadic mass operator is the integral of the transition dyadic of a particle over the position of the particle times the particle number density. Applying the same procedure to all terms in the series (9), we obtain the Dyson equation for the coherent field:

$$\begin{aligned}
\langle \mathbf{E} \rangle &= \mathbf{E}_0 + n_0 \int_D \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \mathbf{E}_0 \, \mathrm{d}^3\mathbf{R}_i \\
&\quad + n_0^2 \int_D \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_j \, \mathbf{E}_0 \, \mathrm{d}^3\mathbf{R}_j \mathrm{d}^3\mathbf{R}_i + \cdots \\
&= \mathbf{E}_0 + \overline{\mathbf{G}}_0 \, \overline{\mathbf{M}} \, \mathbf{E}_0 + \overline{\mathbf{G}}_0 \, \overline{\mathbf{M}} \, \overline{\mathbf{G}}_0 \, \overline{\mathbf{M}} \, \mathbf{E}_0 + \cdots \\
&= \mathbf{E}_0 + \overline{\mathbf{G}}_0 \, \overline{\mathbf{M}} \, \langle \mathbf{E} \rangle \, . \qquad (14)
\end{aligned}$$

With the notation

$$\langle \mathbf{E}(\mathbf{r}) \rangle = {}_{\mathbf{r}}\!\blacktriangleleft\!\!= \stackrel{\mathrm{def}}{=} \blacktriangleleft\!\!=$$

the diagramatic derivation of the Dyson equation (14) is

$$\begin{aligned}
\blacktriangleleft\!\!= &= \Big\langle \blacktriangleleft\!\!- + \sum_i -\!\overset{i}{\circ}\!\blacktriangleleft\!\!- + \sum_{i,j;\, j\neq i} -\!\overset{i}{\circ}\!-\!\overset{j}{\circ}\!\blacktriangleleft\!\!- + \cdots \Big\rangle \\
&= \blacktriangleleft\!\!- + -\Big\langle \sum_i \overset{i}{\circ} \Big\rangle \blacktriangleleft\!\!- + -\Big\langle \sum_{i,j;\, j\neq i} \overset{i}{\circ}\!-\!\overset{j}{\circ} \Big\rangle \blacktriangleleft\!\!- + \cdots \\
&= \blacktriangleleft\!\!- + -\Big\langle \sum_i \overset{i}{\circ} \Big\rangle \blacktriangleleft\!\!- + -\Big\langle \sum_i \overset{i}{\circ} \Big\rangle - \Big\langle \sum_j \overset{j}{\circ} \Big\rangle \blacktriangleleft\!\!- + \cdots \\
&= \blacktriangleleft\!\!- + -\!\odot\!\blacktriangleleft\!\!- + -\!\odot\!-\!\odot\!\blacktriangleleft\!\!- + \cdots \\
&= \blacktriangleleft\!\!- + -\!\odot\!\blacktriangleleft\!\!=, \qquad (15)
\end{aligned}$$

where the dyadic mass operator is represented as

$$\overline{\mathbf{M}}(\mathbf{r}, \mathbf{r}') = \Big\langle \sum_i {}_{\mathbf{r}}\overset{i}{\circ}{}_{\mathbf{r}'} \Big\rangle \stackrel{\mathrm{def}}{=} \Big\langle \sum_i \overset{i}{\circ} \Big\rangle = \odot$$

and, according to the assumption that the positions of the particles are uncorrelated, we used the digramatic computation rule

$$\Big\langle \sum_{i,j;\, j\neq i} \overset{i}{\circ}\!-\!\overset{j}{\circ} \Big\rangle = \Big\langle \sum_i \overset{i}{\circ} \Big\rangle - \Big\langle \sum_j \overset{j}{\circ} \Big\rangle .$$

Employing the same arguments for the Foldy equations for the dyadic Green's function as given by Eqs. (4) and (5), and defining

$$\big\langle \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \big\rangle = {}_{\mathbf{r}}\!=\!{}_{\mathbf{r}'} \stackrel{\mathrm{def}}{=} =,$$

we find

$$\begin{aligned}
= &= \Big\langle - + \sum_i -\!\overset{i}{\circ}\!- + \sum_{i,j;\, j\neq i} -\!\overset{i}{\circ}\!-\!\overset{j}{\circ}\!- + \cdots \Big\rangle \\
&= - + -\!\odot\!=, \qquad (16)
\end{aligned}$$

which is a diagramatic representation of the Dyson equation for the configuration-averaged dyadic Green's function:

$$\langle \overline{\mathbf{G}} \rangle = \overline{\mathbf{G}}_0 + \overline{\mathbf{G}}_0 \, \overline{\mathbf{M}} \, \langle \overline{\mathbf{G}} \rangle. \tag{17}$$

2.2 The Bethe–Salpeter equation for the dyadic correlation function

The Dyson equation for the coherent field has been obtained by taking the configuration average of the total field under the Twersky approximation as well as assuming that the positions of the particles are uncorrelated. In a similar fashion, an equation governing a second-order moment of the field, the so-called Bethe–Salpeter equation, can be obtained.

The derivation of the Bethe–Salpeter equation for the dyadic correlation function under the ladder approximation parallels that described in Ref. [1]. With S being the set of all N random scatterers (i) we consider two disjoint subsets A_0 and B_0 of S, i.e., $A_0 \cap B_0 = \emptyset$, (ii) we fix a particle i, and consider two disjoint subsets A_i and B_i of $S_i = S \setminus \{i\}$, (iii) we fix two particles i and j, and consider two disjoint subsets A_{ij} and B_{ij} of $S_{ij} = S \setminus \{i, j\}$, and so on. The sum of all scattering paths going through the particles in the sets A_0, $A_i \cup \{i\}$, $A_{ij} \cup \{i, j\}$, etc. gives the direct field $\mathbf{E}$, while the sum of all scattering paths going through the particles in the sets B, $B_i \cup \{i\}$, etc. gives the complex conjugate field $\mathbf{E}^\star$. As the sets are disjoint, the dyadic correlation function $\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^\star(\mathbf{r}') \rangle$ computes as

$$\begin{aligned}
&\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^\star(\mathbf{r}') \rangle \\
&= \textstyle\sum_{A_0} \sum_{B_0} \{\mathbf{E}(\mathbf{r})\}_{A_0} \otimes \{\mathbf{E}^\star(\mathbf{r}')\}_{B_0} \\
&+ n_0 \int_D \textstyle\sum_{A_i} \sum_{B_i} \{\mathbf{E}(\mathbf{r})\}_{A_i} \otimes \{\mathbf{E}^\star(\mathbf{r}')\}_{B_i} \, \mathrm{d}^3\mathbf{R}_i \\
&+ n_0^2 \int_D \textstyle\sum_{A_{ij}} \sum_{B_{ij}} \{\mathbf{E}(\mathbf{r})\}_{A_{ij}} \otimes \{\mathbf{E}^\star(\mathbf{r}')\}_{B_{ij}} \, \mathrm{d}^3\mathbf{R}_j \mathrm{d}^3\mathbf{R}_i + \cdots ,
\end{aligned} \tag{18}$$

where $\{\mathbf{E}(\mathbf{r})\}_A$ is the configuration average of the fields corresponding to all self-avoiding paths $\mathcal{P}(A)$ going through the particles in the set A taken over the positions of the particles in the set A. The sums $\sum_{A_0} \sum_{B_0}$ involve all possible realizations of the sets A_0 and B_0, and for each realization, only those pairs of self-avoiding paths $(\mathcal{P}(A_0), \mathcal{P}(B_0))$ which do not appear in previous realizations of A_0 and B_0 are taken into account. In the following, we assume that for large N, we can approximate

$$\textstyle\sum_{A_0} \sum_{B_0} \{\mathbf{E}(\mathbf{r})\}_{A_0} \otimes \{\mathbf{E}^\star(\mathbf{r}')\}_{B_0} \approx \{\mathbf{E}(\mathbf{r})\}_S \otimes \{\mathbf{E}^\star(\mathbf{r}')\}_S. \tag{19}$$

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Applying the above assumption to all terms in the series (18) yields

$$
\begin{aligned}
&\big\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^{\star}(\mathbf{r}') \big\rangle \\
&= \big\{\mathbf{E}(\mathbf{r})\big\}_{S} \otimes \big\{\mathbf{E}^{\star}(\mathbf{r}')\big\}_{S} + n_0 \int_D \big\{\mathbf{E}(\mathbf{r})\big\}_{S_i} \otimes \big\{\mathbf{E}^{\star}(\mathbf{r}')\big\}_{S_i}\, \mathrm{d}^3\mathbf{R}_i \\
&+ n_0^2 \int_D \big\{\mathbf{E}(\mathbf{r})\big\}_{S_{ij}} \otimes \big\{\mathbf{E}^{\star}(\mathbf{r}')\big\}_{S_{ij}}\, \mathrm{d}^3\mathbf{R}_j \mathrm{d}^3\mathbf{R}_i + \cdots .
\end{aligned} \tag{20}
$$

Next we compute the configuration average of the field over the positions of the particles in the sets S_i, S_{ij}, etc., which are the subsets of S with one, two, and more fixed particles. Considering the average field $\{\mathbf{E}(\mathbf{r})\}_{S_i}$, we let S_i^a and S_i^b be two disjoint subsets of S_i, with the property that the paths connecting the observation point $\mathbf{r}$ and particle i go through all particles in the subset S_i^a, and the paths connecting the entrance point (the first particle struck by the incident field) and particle i go through all particles in the subset S_i^b. In computing $\big\{\mathbf{E}(\mathbf{r})\big\}_{S_i}$ we make an assumption which is similar to that in Eq. (19): the configuration average of a field taken over the positions of the particles in a subset A of S is computed by extending the sum over the particles in the subset A to the whole set S. Diagramatically, we have

$$
\begin{aligned}
&\big\{\mathbf{E}(\mathbf{r})\big\}_{S_i} \\
&= \textstyle\sum_{S_i^a}\sum_{S_i^b} \Big\langle \text{—} + \sum\limits_{p\in S_i^a} \text{—}\overset{p}{\circ}\text{—} + \sum\limits_{p,q\in S_i^a;\, q\neq p} \text{—}\overset{p}{\circ}\text{—}\overset{q}{\circ}\text{—} + \cdots \Big\rangle \\
&\overset{i}{\circ}\Big\langle \blacktriangleleft\text{—} + \sum_{n\in S_i^b} \text{—}\overset{n}{\circ}\blacktriangleleft\text{—} + \sum_{n,m\in S_i^b;\, m\neq n} \text{—}\overset{n}{\circ}\text{—}\overset{m}{\circ}\blacktriangleleft\text{—} + \cdots \Big\rangle \\
&= \Big\langle \text{—} + \sum_{p} \text{—}\overset{p}{\circ}\text{—} + \sum_{p,q;\, q\neq p} \text{—}\overset{p}{\circ}\text{—}\overset{q}{\circ}\text{—} + \cdots \Big\rangle \\
&\overset{i}{\circ}\Big\langle \blacktriangleleft\text{—} + \sum_{n} \text{—}\overset{n}{\circ}\blacktriangleleft\text{—} + \sum_{n,m;\, m\neq n} \text{—}\overset{n}{\circ}\text{—}\overset{m}{\circ}\blacktriangleleft\text{—} + \cdots \Big\rangle \\
&= \text{═}\overset{i}{\odot}\blacktriangleleft\text{═},
\end{aligned} \tag{21}
$$

that is,

$$
\{\mathbf{E}(\mathbf{r})\}_{S_i} = \int_{D_i} \big\langle \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}_1) \big\rangle \cdot \overline{\mathbf{T}}_i(\mathbf{r}_1, \mathbf{r}_2) \cdot \big\langle \mathbf{E}(\mathbf{r}_2) \big\rangle \, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2. \tag{22}
$$

Using the dyadic identities

$$
(\overline{\mathbf{A}} \cdot \mathbf{a}) \otimes (\overline{\mathbf{B}} \cdot \mathbf{b}) = (\overline{\mathbf{A}} \otimes \overline{\mathbf{B}}) \cdot (\mathbf{a} \otimes \mathbf{b}), \tag{23}
$$
$$
(\overline{\mathbf{A}} \cdot \overline{\mathbf{C}}) \otimes (\overline{\mathbf{B}} \cdot \overline{\mathbf{D}}) = (\overline{\mathbf{A}} \otimes \overline{\mathbf{B}}) \cdot (\overline{\mathbf{C}} \otimes \overline{\mathbf{D}}), \tag{24}
$$

extending the domain of definition of the transition dyadic $\overline{\mathbf{T}}_i$ to the whole D, and interchanging the order of integrations, we find that the first term in the series (20) is given by

$$
\begin{aligned}
& n_0 \int_D \{\mathbf{E}(\mathbf{r})\}_{S_i} \otimes \{\mathbf{E}^\star(\mathbf{r}')\}_{S_i}\, \mathrm{d}^3\mathbf{R}_i \\
& = n_0 \int_D \Big\{ \int_{D_i} [\langle \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}_1)\rangle \cdot \overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2)] \otimes [\langle \overline{\mathbf{G}}^\star(\mathbf{r}',\mathbf{r}'_1)\rangle \cdot \overline{\mathbf{T}}_i^\star(\mathbf{r}'_1,\mathbf{r}'_2)] \\
& \cdot [\langle \mathbf{E}(\mathbf{r}_2)\rangle \otimes \langle \mathbf{E}^\star(\mathbf{r}'_2)\rangle]\, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}'_1 \mathrm{d}^3\mathbf{r}'_2 \Big\}\, \mathrm{d}^3\mathbf{R}_i \\
& = \int_D [\langle \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}_1)\rangle \otimes \langle \overline{\mathbf{G}}^\star(\mathbf{r}',\mathbf{r}'_1)\rangle] \cdot \overline{\overline{\mathbf{I}}}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2) \\
& \cdot [\langle \mathbf{E}(\mathbf{r}_2)\rangle \otimes \langle \mathbf{E}^\star(\mathbf{r}'_2)\rangle]\, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}'_1 \mathrm{d}^3\mathbf{r}'_2, \qquad (25)
\end{aligned}
$$

where the scattering intensity operator is defined by

$$
\overline{\overline{\mathbf{I}}}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2) = n_0 \int_D \overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2) \otimes \overline{\mathbf{T}}_i^\star(\mathbf{r}'_1,\mathbf{r}'_2)\, \mathrm{d}^3\mathbf{R}_i. \qquad (26)
$$

To compute the configuration average with two fixed particles we proceed analogously. In this case, we find

$$
\{\mathbf{E}(\mathbf{r})\}_{S_{ij}} = \;=\!\overset{i}{\odot}\!=\!\overset{j}{\odot}\!\blacktriangleleft\!=, \qquad (27)
$$

or explicitly,

$$
\begin{aligned}
\{\mathbf{E}(\mathbf{r})\}_{S_{ij}} = \int_{D_i} & \langle \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}_1)\rangle \cdot \overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2) \cdot \langle \overline{\mathbf{G}}(\mathbf{r}_2,\mathbf{r}_3)\rangle \cdot \overline{\mathbf{T}}_j(\mathbf{r}_3,\mathbf{r}_4) \\
& \cdot \langle \mathbf{E}(\mathbf{r}_4)\rangle\, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}_3 \mathrm{d}^3\mathbf{r}_4. \qquad (28)
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
& n_0^2 \int_D \{\mathbf{E}(\mathbf{r})\}_{S_{ij}} \otimes \{\mathbf{E}^\star(\mathbf{r}')\}_{S_{ij}}\, \mathrm{d}^3\mathbf{R}_i \mathrm{d}^3\mathbf{R}_j \\
& = \int_D [\langle \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}_1)\rangle \otimes \langle \overline{\mathbf{G}}^\star(\mathbf{r}',\mathbf{r}'_1)\rangle] \cdot \overline{\overline{\mathbf{I}}}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2) \\
& \cdot [\langle \overline{\mathbf{G}}(\mathbf{r}_2,\mathbf{r}_3)\rangle \otimes \langle \overline{\mathbf{G}}^\star(\mathbf{r}'_2,\mathbf{r}'_3)\rangle] \cdot \overline{\overline{\mathbf{I}}}(\mathbf{r}_3,\mathbf{r}_4,\mathbf{r}'_3,\mathbf{r}'_4) \\
& \cdot [\langle \mathbf{E}(\mathbf{r}_4)\rangle \otimes \langle \mathbf{E}^\star(\mathbf{r}'_4)\rangle]\, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}'_1 \mathrm{d}^3\mathbf{r}'_2\, \mathrm{d}^3\mathbf{r}_3 \mathrm{d}^3\mathbf{r}_4 \mathrm{d}^3\mathbf{r}'_3 \mathrm{d}^3\mathbf{r}'_4. \qquad (29)
\end{aligned}
$$

Note that the ladder approximation for the dyadic correlation function fundamentally relies on the assumption that in Eq. (29), the order of the particles i and j in the expressions of $\{\mathbf{E}\}_{S_{ij}}$ and $\{\mathbf{E}^\star\}_{S_{ij}}$ is the same, or equivalently, that the ordered set of connected particles $\{i, j\}$ is associated with both the direct and the complex-conjugate field. The same is true for the other terms in Eq. (20). By keeping only such ladder contributions and summing them up, we deduce that Eq. (20) is the iterated solution of the integral equation

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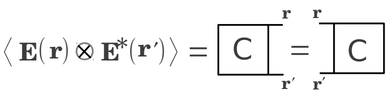
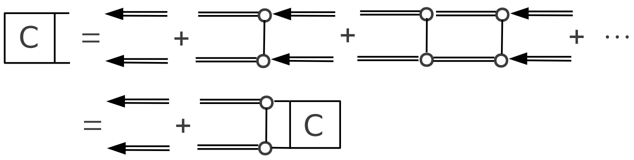
$$
\begin{aligned}
\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^\star(\mathbf{r}') \rangle = {} & \langle \mathbf{E}(\mathbf{r}) \rangle \otimes \langle \mathbf{E}^\star(\mathbf{r}') \rangle \\
& + \int_D \left[\langle \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}_1) \rangle \otimes \langle \overline{\mathbf{G}}^\star(\mathbf{r}', \mathbf{r}'_1) \rangle \right] \cdot \overline{\overline{\mathbf{I}}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}'_1, \mathbf{r}'_2) \\
& \cdot \langle \mathbf{E}(\mathbf{r}_2) \otimes \mathbf{E}^\star(\mathbf{r}'_2) \rangle \, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}'_1 \mathrm{d}^3\mathbf{r}'_2,
\end{aligned}
\tag{30}
$$

with the scattering intensity operator being given by Eq. (26). Eq. (30) is the Bethe–Salpeter equation for the dyadic correlation function.

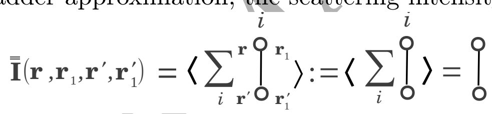
Using the following diagramatic representation for the dyadic correlation function $\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^\star(\mathbf{r}') \rangle$,



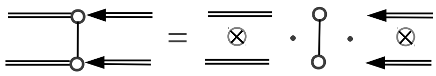
we illustrate the Bethe–Salpeter equation (30) and its iterated solution as



where, in the ladder approximation, the scattering intensity operator is



and for example, the following product rule for the upper and lower scattering path applies



3 Coherent field

The derivation of the Dyson equation (14) is based on the Twersky approximation. An alternative derivation makes use of the Foldy approximation for the exciting fields. To show this, we consider the Foldy equation for the total field (cf. Eq. (2)) $\mathbf{E} = \mathbf{E}_0 + \sum_i \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \mathbf{E}_{\mathsf{exc}i}$. The exciting field $\mathbf{E}_{\mathsf{exc}i}$ is a function of all particle positions, i.e., $\mathbf{E}_{\mathsf{exc}i} = \mathbf{E}_{\mathsf{exc}i}(\mathbf{R}_1, ..., \mathbf{R}_N)$, while the transition dyadic $\overline{\mathbf{T}}_i$ is only a function of the ith particle, i.e, $\overline{\mathbf{T}}_i = \overline{\mathbf{T}}_i(\mathbf{R}_i)$. Taking the configuration average of this equation under the assumption that the *positions of the particles are uncorrelated*, we obtain

$$
\langle \mathbf{E} \rangle = \mathbf{E}_0 + n_0 \int_D \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \langle \mathbf{E}_{\mathsf{exc}i} \rangle_i \, \mathrm{d}^3\mathbf{R}_i,
\tag{31}
$$

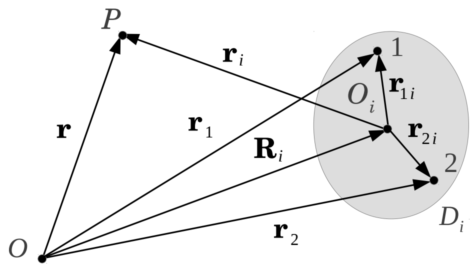


Figure 2: The geometry showing the relevant quantities for computing the integral (35).

where $\langle \mathbf{E}_{\mathtt{exc}i} \rangle_i$ is the conditional probability of the exciting field with the position of particle i held fixed. Employing the Foldy approximation

$$\langle \mathbf{E}_{\mathtt{exc}i} \rangle_i = \langle \mathbf{E} \rangle, \tag{32}$$

the integral equation (31) becomes

$$\langle \mathbf{E} \rangle = \mathbf{E}_0 + n_0 \int_D \overline{\mathbf{G}}_0 \, \overline{\mathbf{T}}_i \, \langle \mathbf{E} \rangle \, \mathrm{d}^3 \mathbf{R}_i. \tag{33}$$

From Eq. (14), we infer that Eq. (33) is the Dyson equation with the dyadic mass operator as in Eq. (13).

The next step is to derive the Foldy integral equation for the coherent field (see [1, 11]). Now, the far-field approximation will come into play. In order to simplify the notation we set, as usual, $\mathbf{E}_{\mathtt{c}}(\mathbf{r}) = \langle \mathbf{E}(\mathbf{r}) \rangle$, and express the Dyson equation (33) in explicit form as

$$\begin{aligned} \mathbf{E}_{\mathtt{c}}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0 \int_D \Big[\int_{D_i} & \overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_1) \cdot \overline{\mathbf{T}}_i(\mathbf{r}_1, \mathbf{r}_2) \\ & \cdot \mathbf{E}_{\mathtt{c}}(\mathbf{r}_2) \, \mathrm{d}^3 \mathbf{r}_1 \mathrm{d}^3 \mathbf{r}_2 \Big] \, \mathrm{d}^3 \mathbf{R}_i. \end{aligned} \tag{34}$$

In Eq. (34) we used the fact that the support of $\overline{\mathbf{T}}_i$ is D_i, so that the integration domain for $\mathbf{r}_1$ and $\mathbf{r}_2$ is D_i. To compute the integral over D_i, we choose the origin of the coordinate system at O_i, and let $\mathbf{r} = \mathbf{r}_i + \mathbf{R}_i$, $\mathbf{r}_1 = \mathbf{r}_{1i} + \mathbf{R}_i$, and $\mathbf{r}_2 = \mathbf{r}_{2i} + \mathbf{R}_i$ (Fig. 2). Then, using $\overline{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_1) = \overline{\mathbf{G}}_0(\mathbf{r}_i, \mathbf{r}_{1i})$ and $\overline{\mathbf{T}}_i(\mathbf{r}_1, \mathbf{r}_2) = \overline{\mathbf{T}}(\mathbf{r}_1 - \mathbf{R}_i, \mathbf{r}_2 - \mathbf{R}_i) = \overline{\mathbf{T}}(\mathbf{r}_{1i}, \mathbf{r}_{2i})$, where $\overline{\mathbf{T}}$ is the transition dyadic of a particle centered at the origin of the coordinate system, we deduce that the integral over D_i is the field scattered by particle i when excited by the coherent field, i.e.,

$$\mathbf{E}_{\mathtt{sct}i}(\mathbf{r}) = \int_{D_i} \overline{\mathbf{G}}_0(\mathbf{r}_i, \mathbf{r}_{1i}) \cdot \overline{\mathbf{T}}(\mathbf{r}_{1i}, \mathbf{r}_{2i}) \cdot \mathbf{E}_{\mathtt{c}}(\mathbf{r}_2) \, \mathrm{d}^3 \mathbf{r}_{1i} \mathrm{d}^3 \mathbf{r}_{2i}. \tag{35}$$

An approximate representation for the coherent field $\mathbf{E}_{\mathtt{c}}(\mathbf{r}_2)$ in D_i, can be obtained in the framework of the characteristic waves method (Appendix 4). In this setting, the coherent field propagating along the incidence direction $\widehat{\mathbf{s}}$ can be written as

$$\mathbf{E}_{\mathtt{c}}(\mathbf{r}) = \sum_{n=1}^{2} \mathrm{e}^{\mathrm{j}K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}\cdot\mathbf{r}} \boldsymbol{\mathfrak{E}}_{n\mathtt{T}}(\widehat{\mathbf{s}}), \tag{36}$$

where $K_n(\widehat{\mathbf{s}})$ is the effective wavenumber, and $\boldsymbol{\mathfrak{E}}_{n\mathtt{T}}(\widehat{\mathbf{s}})$ is the transverse characteristic wave polarization associated with $K_n(\widehat{\mathbf{s}})$ [6]. Then, supposing that for a sparse concentration of particles, $K_n(\widehat{\mathbf{s}}) \approx k_1$, we approximate

$$\begin{aligned} K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}\cdot\mathbf{r}_2 &= K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}\cdot\mathbf{R}_i + K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}\cdot\mathbf{r}_{2i} \\ &\approx K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}\cdot\mathbf{R}_i + k_1\widehat{\mathbf{s}}\cdot\mathbf{r}_{2i}, \end{aligned} \tag{37}$$

implying

$$\mathbf{E}_{\mathtt{c}}(\mathbf{r}_2) \approx \mathrm{e}^{\mathrm{j}k_1\widehat{\mathbf{s}}\cdot\mathbf{r}_{2i}}\mathbf{E}_{\mathtt{c}}(\mathbf{R}_i), \;\; \widehat{\mathbf{s}}\cdot\mathbf{E}_{\mathtt{c}}(\mathbf{R}_i) = 0. \tag{38}$$

Note that this result which states that the coherent field in D_i can be approximated analytically by a plane electromagnetic wave with amplitude $\mathbf{E}_{\mathtt{c}}(\mathbf{R}_i)$, wavenumber k_1, and propagation direction $\widehat{\mathbf{s}}$ has also been established in Ref. [1], provided that the Twersky approximation is used in conjunction with the far-field Foldy equations. Also note that for sparse media, the assumption $K_n(\widehat{\mathbf{s}}) \approx k_1$ is typical of the characteristic waves method (Appendix 4). Furthermore, using the far-field representation for the dyadic Green's function

$$\overline{\mathbf{G}}_0(\mathbf{r}_i,\mathbf{r}_{1i}) = (\overline{\mathbf{I}} - \widehat{\mathbf{r}}_i\otimes\widehat{\mathbf{r}}_i)\frac{\mathrm{e}^{\mathrm{j}k_1 r_i}}{4\pi r_i}\mathrm{e}^{-\mathrm{j}k_1\widehat{\mathbf{r}}_i\cdot\mathbf{r}_{1i}}, \tag{39}$$

and the relation between the far-field scattering dyadic $\overline{\mathbf{A}}$ and the Fourier transform of the transition dyadic $\overline{\mathbf{T}}_p$ (cf. Eq. (129) of Appendix 3),

$$\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{s}})\cdot\boldsymbol{\mathcal{E}}(\widehat{\mathbf{s}}) = \frac{1}{4\pi}(\overline{\mathbf{I}} - \widehat{\mathbf{r}}_i\otimes\widehat{\mathbf{r}}_i)\cdot\overline{\mathbf{T}}_p(k_1\widehat{\mathbf{r}}_i, k_1\widehat{\mathbf{s}})\cdot\boldsymbol{\mathcal{E}}(\widehat{\mathbf{s}}), \tag{40}$$

where in general, $\boldsymbol{\mathcal{E}}(\widehat{\mathbf{s}})$ is a vector field orthogonal to the incidence direction $\widehat{\mathbf{s}}$, i.e., $\widehat{\mathbf{s}}\cdot\boldsymbol{\mathcal{E}}(\widehat{\mathbf{s}}) = 0$, we obtain

$$\begin{aligned} \mathbf{E}_{\mathtt{sct}i}(\mathbf{r}) &= \int_{D_i} \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathbf{E}_{\mathtt{c}}(\mathbf{r}_2)\,\mathrm{d}^3\mathbf{r}_1\mathrm{d}^3\mathbf{r}_2 \\ &= \frac{\mathrm{e}^{\mathrm{j}k_1 r_i}}{r_i}\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{s}})\cdot\mathbf{E}_{\mathtt{c}}(\mathbf{R}_i). \end{aligned} \tag{41}$$

The above result is also valid for any vector field $\mathbf{E}(\mathbf{r}_2)$ which can be approximated analytically by a plane electromagnetic wave in D_i, that is, for $\mathbf{r}_2 \in D_i$, we have $\mathbf{E}(\mathbf{r}_2) = \exp(\mathrm{j}k_1\widehat{\mathbf{s}}\cdot\mathbf{r}_{2i})\mathbf{E}(\mathbf{R}_i)$ with $\widehat{\mathbf{s}}\cdot\mathbf{E}(\mathbf{R}_i) = 0$. Substituting Eq. (41) in Eq. (34), we obtain the Foldy integral equation for the coherent field

$$\mathbf{E}_{\mathtt{c}}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0\int_D g_0(r_i)\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{s}})\cdot\mathbf{E}_{\mathtt{c}}(\mathbf{R}_i)\,\mathrm{d}^3\mathbf{R}_i, \tag{42}$$

where $g_0(r_i) = \exp(\mathrm{j}k_1 r_i)/r_i$.

To solve the Foldy integral equation, we apply the Helmholtz operator $\triangle + k_1^2$. Using $\triangle \mathbf{E}_0(\mathbf{r}) + k_1^2 \mathbf{E}_0(\mathbf{r}) = \mathbf{0}$, where $\mathbf{0}$ is a zero vector, and

$$\triangle g_0(r_i) + k_1^2 g_0(r_i) = -4\pi\delta(\mathbf{r} - \mathbf{R}_i), \tag{43}$$

we get

$$\triangle \mathbf{E}_{\mathsf{c}}(\mathbf{r}) + [k_1^2 \overline{\mathbf{I}} + 4\pi n_0 \overline{\mathbf{A}}(\widehat{\mathbf{s}}, \widehat{\mathbf{s}})] \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}) = \mathbf{0}. \tag{44}$$

Finally, for $n_0 \ll 1$, we approximate

$$k_1^2 \overline{\mathbf{I}} + 4\pi n_0 \overline{\mathbf{A}}(\widehat{\mathbf{s}}, \widehat{\mathbf{s}}) \approx \Big[k_1 \overline{\mathbf{I}} + \frac{2\pi}{k_1} n_0 \overline{\mathbf{A}}(\widehat{\mathbf{s}}, \widehat{\mathbf{s}}) \Big]^2, \tag{45}$$

and infer that the solution to Eq. (44) is

$$\mathbf{E}_{\mathsf{c}}(\mathbf{r}) = \exp\Big\{ \mathrm{j} \Big[k_1 \overline{\mathbf{I}} + \frac{2\pi}{k_1} n_0 \overline{\mathbf{A}}(\widehat{\mathbf{s}}, \widehat{\mathbf{s}}) \Big] s(\mathbf{r}, -\widehat{\mathbf{s}}) \Big\} \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}_A), \tag{46}$$

where $s = s(\mathbf{r}, -\widehat{\mathbf{s}}) = \widehat{\mathbf{s}} \cdot (\mathbf{r} - \mathbf{r}_A)$, and $\mathbf{r}_A$ is the point where the straight line with the direction vector $-\widehat{\mathbf{s}}$ going through the observation point crosses the lower boundary of the layer. This result coincides with that obtained in Ref. [1].

In the scalar case, an alternative method for solving the Foldy integral equation (42) can be found in Ref. [11]. In the vector case, an adapted version of this more "cumbersome" solution method is given in Appendix 5. Instead of solving the Foldy integral equation, the coherent field can be computed in the framework of the characteristic wave method discussed in Appendix 4. In this approach, the dispersion equation for the effective wavenumber as well as a general representation for the coherent field are formulated in terms of the dyadic mass operator.

4 Second-order moment of the electromagnetic field

To obtain the vector radiative transfer equation we will use the ladder approximated Bethe–Salpeter equation for the dyadic correlation function, or more specifically, an iterated solution of this integral equation. Since we intend to use the far-field approximation from now on, we will first establish some auxiliary results involving integrals of the average dyadic Green's function in the far-field region. Actually, for a fixed particle i, we will compute the integral

$$\int_{D_i} \big\langle \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}_1) \big\rangle \cdot \overline{\mathbf{T}}_i(\mathbf{r}_1, \mathbf{r}_2) \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}_2)\, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2, \tag{47}$$

when the observation point $\mathbf{r}$ is in the far-field region of D_i.

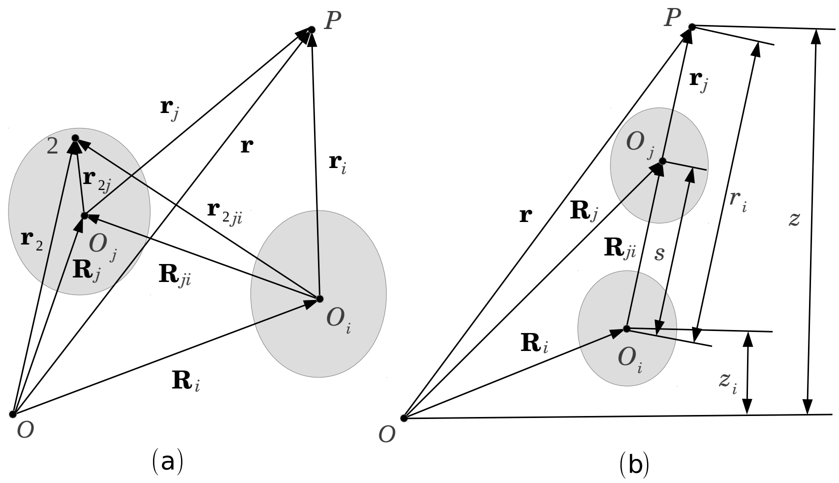


Figure 3: Geometries showing the relevant quantities in (a) solving Eq. (51), and (b) computing the integral (63) by the stationary phase method.

4.1 Integrals of the average dyadic Green's function in the far-field region

The Dyson equation for the average dyadic Green's function is (cf. Eq. (17))

$$\begin{aligned}\langle\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\rangle = \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}') + n_0 \int_D \Big[\int_{D_j} \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{T}}_j(\mathbf{r}_1,\mathbf{r}_2) \\ \cdot\langle\overline{\mathbf{G}}(\mathbf{r}_2,\mathbf{r}')\rangle\,\mathrm{d}^3\mathbf{r}_1\mathrm{d}^3\mathbf{r}_2\Big]\,\mathrm{d}^3\mathbf{R}_j.\end{aligned} \tag{48}$$

For a fixed i, we right-multiply Eq. (48) by $\overline{\mathbf{T}}_i(\mathbf{r}',\mathbf{r}'')\cdot\mathbf{E}_\mathrm{c}(\mathbf{r}'')$ with $\mathbf{r}',\mathbf{r}''\in D_i$, and integrate over $\mathbf{r}'$ and $\mathbf{r}''$. Taking into account that the support of $\overline{\mathbf{T}}_i$ is D_i and that of $\overline{\mathbf{T}}_j$ is D_j, and interchanging the order of integration, we obtain the integral equation

$$\begin{aligned}&\int_{D_i}\langle\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\rangle\cdot\overline{\mathbf{T}}_i(\mathbf{r}',\mathbf{r}'')\cdot\mathbf{E}_\mathrm{c}(\mathbf{r}'')\,\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r}'' \\ &= \int_{D_i}\overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}')\cdot\overline{\mathbf{T}}_i(\mathbf{r}',\mathbf{r}'')\cdot\mathbf{E}_\mathrm{c}(\mathbf{r}'')\,\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r}'' \\ &+ n_0\int_D\Big\{\int_{D_j}\overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{T}}_j(\mathbf{r}_1,\mathbf{r}_2) \\ &\cdot\Big[\int_{D_i}\langle\overline{\mathbf{G}}(\mathbf{r}_2,\mathbf{r}')\rangle\cdot\overline{\mathbf{T}}_i(\mathbf{r}',\mathbf{r}'')\cdot\mathbf{E}_\mathrm{c}(\mathbf{r}'')\,\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r}''\Big]\,\mathrm{d}^3\mathbf{r}_1\mathrm{d}^3\mathbf{r}_2\Big\}\,\mathrm{d}^3\mathbf{R}_j.\end{aligned} \tag{49}$$

To solve Eq. (49) for $\langle \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \rangle$ when the source point $\mathbf{r}'$ is in D_i and the observation point $\mathbf{r}$ is in the far-field region of D_i, we seek a solution in the form

$$\langle \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \rangle = \overline{\mathbf{X}}(\mathbf{r},\mathbf{R}_i) \cdot \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}'). \tag{50}$$

Thus, we assume that the unknown dyadic $\overline{\mathbf{X}}(\mathbf{r},\mathbf{R}_i)$ depends on the observation point $\mathbf{r}$ and the particle position $\mathbf{R}_i$, but not on the source point $\mathbf{r}'$. Moreover, we suppose that $\overline{\mathbf{X}}(\mathbf{r},\mathbf{R}_i)$ is a transverse dyadic with respect to the direction $\widehat{\mathbf{r}}_i$, that is, $\widehat{\mathbf{r}}_i \cdot \overline{\mathbf{X}}(\mathbf{r},\mathbf{R}_i) = \mathbf{0}$ with $\mathbf{r}_i = \mathbf{r} - \mathbf{R}_i$. Inserting Eq. (50) in Eq. (49) gives

$$\begin{aligned}
&\overline{\mathbf{X}}(\mathbf{r},\mathbf{R}_i) \cdot \int_{D_i} \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}') \cdot \overline{\mathbf{T}}_i(\mathbf{r}',\mathbf{r}'') \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}'')\,\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r}'' \\
&= \int_{D_i} \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}') \cdot \overline{\mathbf{T}}_i(\mathbf{r}',\mathbf{r}'') \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}'')\,\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r}'' \\
&+ n_0 \int_D \Big\{ \int_{D_j} \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1) \cdot \overline{\mathbf{T}}_j(\mathbf{r}_1,\mathbf{r}_2) \cdot \overline{\mathbf{X}}(\mathbf{r}_2,\mathbf{R}_i) \\
&\cdot \Big[\int_{D_i} \overline{\mathbf{G}}_0(\mathbf{r}_2,\mathbf{r}') \cdot \overline{\mathbf{T}}_i(\mathbf{r}',\mathbf{r}'') \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}'')\,\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r}'' \Big] \mathrm{d}^3\mathbf{r}_1\mathrm{d}^3\mathbf{r}_2 \Big\}\, \mathrm{d}^3\mathbf{R}_j,
\end{aligned} \tag{51}$$

and we solve now the integral equation (51) for $\overline{\mathbf{X}}(\mathbf{r},\mathbf{R}_i)$. The geometry showing the relevant quantities is illustrated in Fig. 3. The integrals in Eq. (51) are computed as follows. First, we evaluate the second term on the right-hand side of Eq. (51). The integral over D_i, denoted by $\mathbf{E}_i(\mathbf{r}_2)$, is computed by means of Eq. (41); the result is

$$\begin{aligned}
\mathbf{E}_i(\mathbf{r}_2) &= \int_{D_i} \overline{\mathbf{G}}_0(\mathbf{r}_2,\mathbf{r}') \cdot \overline{\mathbf{T}}_i(\mathbf{r}',\mathbf{r}'') \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}'')\,\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r}'' \\
&= \frac{\mathrm{e}^{\mathrm{j}k_1 r_{2ji}}}{r_{2ji}} \overline{\mathbf{A}}(\widehat{\mathbf{r}}_{2ji},\widehat{\mathbf{s}}) \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{R}_i),
\end{aligned} \tag{52}$$

whence, approximating

$$\frac{\mathrm{e}^{\mathrm{j}k_1 r_{2ji}}}{r_{2ji}} \approx \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{R}}_{ji} \cdot \mathbf{r}_{2j}} \frac{\mathrm{e}^{\mathrm{j}k_1 R_{ji}}}{R_{ji}} \tag{53}$$

and

$$\overline{\mathbf{A}}(\widehat{\mathbf{r}}_{2ji},\widehat{\mathbf{s}}) \approx \overline{\mathbf{A}}(\widehat{\mathbf{R}}_{ji},\widehat{\mathbf{s}}), \tag{54}$$

we find that in the coordinate system centered at O_j,

$$\mathbf{E}_i(\mathbf{r}_2) = \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{R}}_{ji} \cdot \mathbf{r}_{2j}} \mathbf{E}_i(\mathbf{R}_j) \tag{55}$$

is a locally plane electromagnetic wave with propagation direction $\widehat{\mathbf{R}}_{ji}$ and amplitude

$$\mathbf{E}_i(\mathbf{R}_j) = \frac{\mathrm{e}^{\mathrm{j}k_1 R_{ji}}}{R_{ji}} \overline{\mathbf{A}}(\widehat{\mathbf{R}}_{ji},\widehat{\mathbf{s}}) \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{R}_i). \tag{56}$$

For computing the integral over D_j, we approximate

$$\overline{\mathbf{X}}(\mathbf{r}_2, \mathbf{R}_i) \approx \overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i), \tag{57}$$

and define

$$\mathbf{E}_j(\mathbf{r}_2) = \overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i) \cdot \mathbf{E}_i(\mathbf{r}_2) = \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{R}}_{ji} \cdot \mathbf{r}_{2j}} \mathbf{E}_j(\mathbf{R}_j), \tag{58}$$

with

$$\mathbf{E}_j(\mathbf{R}_j) = \overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i) \cdot \mathbf{E}_i(\mathbf{R}_j). \tag{59}$$

Thus, analogously to $\mathbf{E}_i(\mathbf{r}_2)$, $\mathbf{E}_j(\mathbf{r}_2)$ is also a plane electromagnetic wave with propagation direction $\widehat{\mathbf{R}}_{ji}$ and an amplitude as in Eq. (59). Noting the orthogonality relation

$$\widehat{\mathbf{R}}_{ji} \cdot \mathbf{E}_j(\mathbf{R}_j) = \widehat{\mathbf{R}}_{ji} \cdot [\overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i) \cdot \mathbf{E}_i(\mathbf{R}_j)] = 0, \tag{60}$$

owing to $\widehat{\mathbf{R}}_{ji} \cdot \overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i) = \mathbf{0}$, and applying again Eq. (41) with $\mathbf{E}_j(\mathbf{r}_2)$ in place of $\mathbf{E}_\mathsf{c}(\mathbf{r}_2)$, we find that the second term on the right-hand side of Eq. (51) is

$$n_0 \int_D \frac{\mathrm{e}^{\mathrm{j}k_1 r_j}}{r_j} \overline{\mathbf{A}}(\widehat{\mathbf{r}}_j, \widehat{\mathbf{R}}_{ji}) \cdot \mathbf{E}_j(\mathbf{R}_j)\, \mathrm{d}^3\mathbf{R}_j. \tag{61}$$

For the integrals over D_i on the left-hand side of Eq. (51) and in the first term on the right-hand side of Eq. (51), we directly apply Eq. (41); taking account of Eqs. (56), (59), and (61), we end up with

$$\begin{aligned} &\overline{\mathbf{X}}(\mathbf{r}, \mathbf{R}_i) \cdot \Big[\frac{\mathrm{e}^{\mathrm{j}k_1 r_i}}{r_i} \overline{\mathbf{A}}(\widehat{\mathbf{r}}_i, \widehat{\mathbf{s}}) \cdot \mathbf{E}_\mathsf{c}(\mathbf{R}_i)\Big] \\ &\quad = \frac{\mathrm{e}^{\mathrm{j}k_1 r_i}}{r_i} \overline{\mathbf{A}}(\widehat{\mathbf{r}}_i, \widehat{\mathbf{s}}) \cdot \mathbf{E}_\mathsf{c}(\mathbf{R}_i) + n_0 \int_D \frac{\mathrm{e}^{\mathrm{j}k_1 r_j}}{r_j} \overline{\mathbf{A}}(\widehat{\mathbf{r}}_j, \widehat{\mathbf{R}}_{ji}) \\ &\quad \cdot \overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i) \cdot \frac{\mathrm{e}^{\mathrm{j}k_1 R_{ji}}}{R_{ji}} \overline{\mathbf{A}}(\widehat{\mathbf{R}}_{ji}, \widehat{\mathbf{s}}) \cdot \mathbf{E}_\mathsf{c}(\mathbf{R}_i)\, \mathrm{d}^3\mathbf{R}_j. \end{aligned} \tag{62}$$

The integral in Eq. (62) is computed by the stationary phase method [11]. Referring to Fig. 3b, we obtain

$$\begin{aligned} &\int_D \frac{\mathrm{e}^{\mathrm{j}k_1 r_j}}{r_j} \overline{\mathbf{A}}(\widehat{\mathbf{r}}_j, \widehat{\mathbf{R}}_{ji}) \cdot \overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i) \cdot \frac{\mathrm{e}^{\mathrm{j}k_1 R_{ji}}}{R_{ji}} \overline{\mathbf{A}}(\widehat{\mathbf{R}}_{ji}, \widehat{\mathbf{s}}) \cdot \mathbf{E}_\mathsf{c}(\mathbf{R}_i)\, \mathrm{d}^3\mathbf{R}_j \\ &\quad = \frac{\mathrm{e}^{\mathrm{j}k_1 r_i}}{r_i} \mathrm{j}\frac{2\pi}{k_1} \overline{\mathbf{A}}(\widehat{\mathbf{r}}_i, \widehat{\mathbf{r}}_i) \cdot \Big[\frac{1}{\widehat{\mathbf{r}}_i \cdot \widehat{\mathbf{z}}} \int_{z_i}^{z} \overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i)\, \mathrm{d}z_j\Big] \cdot \overline{\mathbf{A}}(\widehat{\mathbf{r}}_i, \widehat{\mathbf{s}}) \cdot \mathbf{E}_\mathsf{c}(\mathbf{R}_i), \end{aligned} \tag{63}$$

and we arrive at the integral equation

$$\overline{\mathbf{X}}(\mathbf{r}, \mathbf{R}_i) = \overline{\mathbf{I}} + \mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{A}}(\widehat{\mathbf{r}}_i, \widehat{\mathbf{r}}_i) \cdot \Big[\frac{1}{\widehat{\mathbf{r}}_i \cdot \widehat{\mathbf{z}}} \int_{z_i}^{z} \overline{\mathbf{X}}(\mathbf{R}_j, \mathbf{R}_i)\, \mathrm{d}z_j\Big]. \tag{64}$$

It is not hard to see that the solution of this integral equation is

$$\overline{\mathbf{X}}(\mathbf{r},\mathbf{R}_i)=\exp\Big[\mathrm{j}\frac{2\pi}{k_1}n_0\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{r}}_i)|\mathbf{r}-\mathbf{R}_i|\Big]=\exp\Big[\mathrm{j}\frac{2\pi}{k_1}n_0\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{r}}_i)r_i\Big]. \tag{65}$$

Indeed, for

$$\overline{\mathbf{W}}=\mathrm{j}\frac{2\pi}{k_1}n_0\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{r}}_i), \tag{66}$$

the results

$$\overline{\mathbf{X}}(\mathbf{r},\mathbf{R}_i)=\exp(\overline{\mathbf{W}}|\mathbf{r}-\mathbf{R}_i|)=\exp(\overline{\mathbf{W}}r_i), \tag{67}$$

$$\overline{\mathbf{X}}(\mathbf{R}_j,\mathbf{R}_i)=\exp(\overline{\mathbf{W}}|\mathbf{R}_j-\mathbf{R}_i|)=\exp(\overline{\mathbf{W}}s), \tag{68}$$

together with the dyadic identity

$$\mathrm{e}^{\overline{\mathbf{W}}r_i}=\overline{\mathbf{I}}+\overline{\mathbf{W}}\cdot\Big[\int_0^{r_i}\mathrm{e}^{\overline{\mathbf{W}}s}\,\mathrm{d}s\Big] \tag{69}$$

prove the assertion. In Eqs. (68) and (69), s is defined by $s=|\mathbf{R}_j-\mathbf{R}_i|$, and in deriving Eq. (69), we made the change of variable $z_j=z_i+s\widehat{\mathbf{r}}_i\cdot\widehat{\mathbf{z}}$. Hence, from Eqs. (50) and (65), we infer that the desired expression for the average dyadic Green's function is

$$\big\langle\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\big\rangle=\exp\Big[\mathrm{j}\frac{2\pi}{k_1}n_0\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{r}}_i)r_i\Big]\cdot\overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}'),\ \ \mathbf{r}'\in D_i. \tag{70}$$

Combining Eqs. (41) and (70), we find that the integral (47) is

$$\begin{aligned}&\int_{D_i}\big\langle\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}_1)\big\rangle\cdot\overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathbf{E}_{\mathsf{c}}(\mathbf{r}_2)\,\mathrm{d}^3\mathbf{r}_1\mathrm{d}^3\mathbf{r}_2\\&=\frac{\overline{\mathbf{t}}(\widehat{\mathbf{r}}_i,r_i)}{r_i}\cdot\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{s}})\cdot\mathbf{E}_{\mathsf{c}}(\mathbf{R}_i),\end{aligned} \tag{71}$$

where

$$\overline{\mathbf{t}}(\widehat{\mathbf{r}}_i,r_i)=\mathrm{e}^{\mathrm{j}k_1r_i}\exp\Big[\mathrm{j}\frac{2\pi}{k_1}n_0\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{r}}_i)r_i\Big] \tag{72}$$

is the coherent transmission dyadic from particle i to the observation point $\mathbf{r}$. An interesting remark is in order: while the integral in Eq. (41) represents the field scattered by particle i at $\mathbf{r}$, the integral in Eq. (71) represent the field scattered by particle i at $\mathbf{r}$ via various particle sequences.

A further consequence of Eqs. (41) and (70) is the integral result

$$\begin{aligned}&\int_{D_i}\big\langle\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}_1)\big\rangle\cdot\overline{\mathbf{T}}_i(\mathbf{r}_1,\mathbf{r}_2)\\&\cdot\Big[\int_{D_j}\big\langle\overline{\mathbf{G}}(\mathbf{r}_2,\mathbf{r}_3)\big\rangle\cdot\overline{\mathbf{T}}_j(\mathbf{r}_3,\mathbf{r}_4)\cdot\mathbf{E}_{\mathsf{c}}(\mathbf{r}_4)\,\mathrm{d}^3\mathbf{r}_3\mathrm{d}^3\mathbf{r}_4\Big]\mathrm{d}^3\mathbf{r}_1\mathrm{d}^3\mathbf{r}_2\\&=\frac{\overline{\mathbf{t}}(\widehat{\mathbf{r}}_i,r_i)}{r_i}\cdot\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{R}}_{ij})\cdot\frac{\overline{\mathbf{t}}(\widehat{\mathbf{R}}_{ij},R_{ij})}{R_{ji}}\cdot\overline{\mathbf{A}}(\widehat{\mathbf{R}}_{ij},\widehat{\mathbf{s}})\cdot\mathbf{E}_{\mathsf{c}}(\mathbf{R}_j),\end{aligned} \tag{73}$$

which represents the field scattered from j to i to $\mathbf{r}$ via various particle sequences. Note that in deriving Eqs. (73), we used the approximations (53) and (54) as well as the orthogonality relation $\widehat{\mathbf{R}}_{ij} \cdot \bar{\mathbf{t}}(\widehat{\mathbf{R}}_{ij}, R_{ij}) = \mathbf{0}$, which follows from

$$\widehat{\mathbf{R}}_{ij} \cdot \exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{A}}(\widehat{\mathbf{R}}_{ij}, \widehat{\mathbf{R}}_{ij}) R_{ij}\Big] = \mathbf{0}. \tag{74}$$

4.2 Vector radiative transfer equation

The scattering intensity operator given by Eq. (26) is the kernel of the ladder approximated Bethe–Salpeter equation for the dyadic correlation function. Using the dyadic identities

$$(\overline{\mathbf{A}} \otimes \overline{\mathbf{B}}) \cdot (\overline{\mathbf{C}} \otimes \overline{\mathbf{D}}) = (\overline{\mathbf{A}} \cdot \overline{\mathbf{C}}) \otimes (\overline{\mathbf{B}} \cdot \overline{\mathbf{D}}),$$
$$(\overline{\mathbf{A}} \otimes \overline{\mathbf{B}}) \cdot \overline{\mathbf{C}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{C}} \cdot \overline{\mathbf{B}}^T,$$

where T stands for "transposed", we express the Bethe–Salpeter equation (30) as

$$\begin{aligned}
&\big\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^\star(\mathbf{r}')\big\rangle \\
&= \mathbf{E}_\mathsf{c}(\mathbf{r}) \otimes \mathbf{E}_\mathsf{c}^\star(\mathbf{r}') + n_0 \int_D \Big\{ \int_{D_i} [\big\langle \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}_1)\big\rangle \cdot \overline{\mathbf{T}}_i(\mathbf{r}_1, \mathbf{r}_2)] \\
&\cdot \big\langle \mathbf{E}(\mathbf{r}_2) \otimes \mathbf{E}^\star(\mathbf{r}_2')\big\rangle \cdot [\big\langle \overline{\mathbf{G}}^\star(\mathbf{r}', \mathbf{r}_1')\big\rangle \cdot \overline{\mathbf{T}}_i^\star(\mathbf{r}_1', \mathbf{r}_2')]^T \\
&\times \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}_1' \mathrm{d}^3\mathbf{r}_2' \Big\} \,\mathrm{d}^3\mathbf{R}_i,
\end{aligned} \tag{75}$$

and further, upon introducing the dyadic

$$\overline{\mathbf{P}}_i(\mathbf{r}, \mathbf{r}') = \int_{D_i} \big\langle \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}_1)\big\rangle \cdot \overline{\mathbf{T}}_i(\mathbf{r}_1, \mathbf{r}')\,\mathrm{d}^3\mathbf{r}_1, \tag{76}$$

as

$$\begin{aligned}
&\big\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^\star(\mathbf{r}')\big\rangle \\
&= \mathbf{E}_\mathsf{c}(\mathbf{r}) \otimes \mathbf{E}_\mathsf{c}^\star(\mathbf{r}') + n_0 \int_D \Big[\int_{D_i} \overline{\mathbf{P}}_i(\mathbf{r}, \mathbf{r}_2) \cdot \big\langle \mathbf{E}(\mathbf{r}_2) \otimes \mathbf{E}^\star(\mathbf{r}_2')\big\rangle \\
&\cdot \overline{\mathbf{P}}_i^\dagger(\mathbf{r}', \mathbf{r}_2')\,\mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}_2' \Big]\,\mathrm{d}^3\mathbf{R}_i,
\end{aligned} \tag{77}$$

where $\dagger$ stands for conjugate transpose. Iterating Eq. (77) gives

$$\begin{aligned}
&\big\langle \mathbf{E}(\mathbf{r}) \otimes \mathbf{E}^\star(\mathbf{r}')\big\rangle \\
&= \mathbf{E}_\mathsf{c}(\mathbf{r}) \otimes \mathbf{E}_\mathsf{c}^\star(\mathbf{r}') + n_0 \int_D \Big\{ \int_{D_i} \overline{\mathbf{P}}_i(\mathbf{r}, \mathbf{r}_2) \cdot [\mathbf{E}_\mathsf{c}(\mathbf{r}_2) \otimes \mathbf{E}_\mathsf{c}^\star(\mathbf{r}_2')] \\
&\cdot \overline{\mathbf{P}}_i^\dagger(\mathbf{r}', \mathbf{r}_2')\,\mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}_2' \Big\}\,\mathrm{d}^3\mathbf{R}_i \\
&+ n_0^2 \int_D \Big\{ \int_{D_i} \int_{D_j} \overline{\mathbf{P}}_i(\mathbf{r}, \mathbf{r}_2) \cdot \overline{\mathbf{P}}_j(\mathbf{r}_2, \mathbf{r}_4) \cdot [\mathbf{E}_\mathsf{c}(\mathbf{r}_4) \otimes \mathbf{E}_\mathsf{c}^\star(\mathbf{r}_4')] \\
&\cdot \overline{\mathbf{P}}_j^\dagger(\mathbf{r}_2', \mathbf{r}_4') \cdot \overline{\mathbf{P}}_i^\dagger(\mathbf{r}', \mathbf{r}_2') \mathrm{d}^3\mathbf{r}_4 \mathrm{d}^3\mathbf{r}_4' \mathrm{d}^3\mathbf{r}_2 \mathrm{d}^3\mathbf{r}_2' \Big\}\,\mathrm{d}^3\mathbf{R}_j \mathrm{d}^3\mathbf{R}_i + \cdots,
\end{aligned} \tag{78}$$

where the integration domain over $\mathbf{r}_2$ and $\mathbf{r}'_2$ is D_i, while the integration domain over $\mathbf{r}_4$ and $\mathbf{r}'_4$ is D_j.

From Eqs. (71) and (73), we have the integral results

$$\int_{D_i} \overline{\mathbf{P}}_i(\mathbf{r},\mathbf{r}_2)\cdot\mathbf{E}_\mathsf{c}(\mathbf{r}_2)\,\mathrm{d}^3\mathbf{r}_2 = \overline{\mathbf{V}}(\mathbf{r}_i,\widehat{\mathbf{s}})\cdot\mathbf{E}_\mathsf{c}(\mathbf{R}_i) \tag{79}$$

and

$$\begin{aligned}&\int_{D_i}\int_{D_j} \overline{\mathbf{P}}_i(\mathbf{r},\mathbf{r}_2)\cdot\overline{\mathbf{P}}_j(\mathbf{r}_2,\mathbf{r}_4)\cdot\mathbf{E}_\mathsf{c}(\mathbf{r}_4)\,\mathrm{d}^3\mathbf{r}_4\mathrm{d}^3\mathbf{r}_2\\ &= \overline{\mathbf{V}}(\mathbf{r}_i,\widehat{\mathbf{R}}_{ij})\cdot\overline{\mathbf{V}}(\mathbf{R}_{ij},\widehat{\mathbf{s}})\cdot\mathbf{E}_\mathsf{c}(\mathbf{R}_j),\end{aligned} \tag{80}$$

where

$$\overline{\mathbf{V}}(\mathbf{r}_i,\widehat{\mathbf{s}}) = \frac{\overline{\mathbf{t}}(\widehat{\mathbf{r}}_i,r_i)}{r_i}\cdot\overline{\mathbf{A}}(\widehat{\mathbf{r}}_i,\widehat{\mathbf{s}}). \tag{81}$$

Applying now the dyadic identity

$$\overline{\mathbf{A}}\cdot(\mathbf{a}\otimes\mathbf{b})\cdot\overline{\mathbf{B}}^T = (\overline{\mathbf{A}}\cdot\mathbf{a})\otimes(\overline{\mathbf{B}}\cdot\mathbf{b}),$$

setting $\mathbf{r}=\mathbf{r}'$ in Eq. (78), switching to the (ladder) coherency dyadic $\overline{\mathbf{C}}_\mathrm{L}(\mathbf{r}) = \left\langle\mathbf{E}(\mathbf{r})\otimes\mathbf{E}^\star(\mathbf{r})\right\rangle$ and its coherent part $\overline{\mathbf{C}}_\mathsf{c}(\mathbf{r}) = \mathbf{E}_\mathsf{c}(\mathbf{r})\otimes\mathbf{E}^\star_\mathsf{c}(\mathbf{r})$, and using Eqs. (79) and (80), we arrive at

$$\begin{aligned}\overline{\mathbf{C}}_\mathrm{L}(\mathbf{r}) &= \overline{\mathbf{C}}_\mathsf{c}(\mathbf{r}) + n_0\int_D \overline{\mathbf{V}}(\mathbf{r}_i,\widehat{\mathbf{s}})\cdot\overline{\mathbf{C}}_\mathsf{c}(\mathbf{R}_i)\cdot\overline{\mathbf{V}}^\dagger(\mathbf{r}_i,\widehat{\mathbf{s}})\,\mathrm{d}^3\mathbf{R}_i\\ &+ n_0^2\int_D \overline{\mathbf{V}}(\mathbf{r}_i,\widehat{\mathbf{R}}_{ij})\cdot\overline{\mathbf{V}}(\mathbf{R}_{ij},\widehat{\mathbf{s}})\cdot\overline{\mathbf{C}}_\mathsf{c}(\mathbf{R}_j)\cdot\overline{\mathbf{V}}^\dagger(\mathbf{R}_{ij},\widehat{\mathbf{s}})\\ &\cdot\overline{\mathbf{V}}^\dagger(\mathbf{r}_i,\widehat{\mathbf{R}}_{ij})\,\mathrm{d}^3\mathbf{R}_i\mathrm{d}^3\mathbf{R}_j + \cdots.\end{aligned} \tag{82}$$

This expansion was also derived as Eq. (145) in Ref. [1] and was the key point of the further development. The complete derivation of the radiative transfer equation from the expansion (82) was given in Section 11.3 of Ref. [1] and will not be repeated here. Essentially, from Eq. (82), a series expansion for the (ladder) specific coherency dyadic $\overline{\mathbf{\Sigma}}_\mathrm{L}$, defined through the angular spectrum representation

$$\overline{\mathbf{C}}_\mathrm{L}(\mathbf{r}) = \int \overline{\mathbf{\Sigma}}_\mathrm{L}(\mathbf{r},-\widehat{\mathbf{p}})\,\mathrm{d}^2\widehat{\mathbf{p}}, \tag{83}$$

was obtained. This series expansion is the expanded form of an integral equation for $\overline{\mathbf{\Sigma}}_\mathrm{L}$, which can be transformed into an integral equation for the diffuse specific coherency dyadic $\overline{\boldsymbol{\Sigma}}_\mathsf{dL}$, defined by

$$\overline{\mathbf{\Sigma}}_\mathrm{L}(\mathbf{r},-\widehat{\mathbf{p}}) = \overline{\boldsymbol{\Sigma}}_\mathsf{dL}(\mathbf{r},-\widehat{\mathbf{p}}) + \delta(\widehat{\mathbf{p}}+\widehat{\mathbf{s}})\overline{\mathbf{C}}_\mathsf{c}(\mathbf{r}). \tag{84}$$

Finally, differentiating the integral equation for $\overline{\boldsymbol{\Sigma}}_{\mathsf{dL}}$ the vector radiative transfer equation

$$\frac{\mathrm{d}\mathsf{I}_{\mathsf{d}}(\mathbf{r},\widehat{\mathbf{q}})}{\mathrm{d}s} = -n_0\mathsf{K}(\widehat{\mathbf{q}})\mathsf{I}_{\mathsf{d}}(\mathbf{r},\widehat{\mathbf{q}}) + n_0\mathsf{Z}(\widehat{\mathbf{q}},\widehat{\mathbf{s}})\mathsf{I}_{\mathsf{c}}(\mathbf{r}) + n_0\int \mathsf{Z}(\widehat{\mathbf{q}},\widehat{\mathbf{q}}')\mathsf{I}_{\mathsf{d}}(\mathbf{r},\widehat{\mathbf{q}}')\,\mathrm{d}^2\widehat{\mathbf{q}}', \tag{85}$$

is derived. Here, I_{d} is the diffuse specific intensity column vector, I_{c} is the Stokes column vector of the coherent field, while Z and K are the phase and the extinction matrix of a nonspherical particle in a fixed orientation.

5 Conclusions

As in Part I [1], in this paper we have derived the vector radiative transfer equation for a discrete random layer with non-scattering boundaries and a sparse concentration of randomly positioned particles. This time, however, we maximally delayed the use of the far-field assumption for sparsely distributed particles by using the exact integral Foldy equations. Specifically, we proceed as follows.

First, by assuming that the discrete random medium in question if fully ergodic while the positions of the constituent particles are statistically independent, and by applying the Twersky approximation to the iterated solution of the integral Foldy equations, we derived the Dyson equation for the configuration-averaged coherent field and the ladder approximated Bethe–Salpeter equation for the configuration-averaged dyadic correlation function. Furthermore, we proved that the Dyson equation can be also obtained by means of the Foldy approximation for the exciting fields. Strictly speaking, these results do not rely on the electromagnetic far-field assumption and can thus be considered rather general. One can argue however that the particle number density should be sufficiently low to ensure the presumed statistical independence and uniformity of particle positions.

Second, we showed that under the far-field approximation, the Dyson equation reduces to the Foldy integral equation for the coherent field. To this end, we used the far-field representation of the dyadic Green's function and the relation between the far-field scattering dyadic and the Fourier transform of the transition dyadic. We discussed two methods for solving the Foldy integral equation: an approach based on the application of the Helmholtz operator to the integral equation, and an approach similar to that used by Ishimaru in the scalar case [11]. Next, we showed that the same expression for the coherent field can be obtained in the framework of the characteristic waves method. However, in this case, the additional assumptions that the coherent field propagates along the incidence direction and that the effective wavenumber is close to that of the background medium had to be imposed.

Third, we derived the vector radiative transfer equation by considering an iterated solution of the Bethe–Salpeter equation for the dyadic correlation function and by computing each term in the series under the far-field approximation.

In fact, we estimated various integrals involving the average dyadic Green's function in the far-field region. Switching to the coherency dyadic, we found that the resulting series representation coincides with that obtained in Ref. [1]. A direct consequence of this series expansion is an integral equation for the diffuse specific coherency dyadic implying the vector radiative transfer equation for the diffuse specific intensity column vector.

In the final analysis, the same assumptions and approximations as those used in Ref. [1] have been invoked, albeit in a different order, and the same vector radiative transfer equation has been arrived at. However, two important comments are called for.

First, it should be pointed out that a key point of the analysis in this paper is the assumption (38) stating that the coherent field can be locally approximated by a plane electromagnetic wave having the same wavenumber as that of the background medium. This assumption implies the relation (41) which is used many times during the derivation. To justify the assumption (38) we used two arguments which, strictly speaking, are not necessarily consistent with the present approach: one is a result of the characteristic wave method, wherein the assumptions mentioned above are made, and the other one is a result of the analysis performed in Ref. [1], wherein the Twersky approximation is used in conjunction with the far-field Foldy equations.

Second, it is relatively straightforward to justify the use of the ladder approximation in the computation of second moments in the field after having made the far-field assumption. Indeed, in that case different multi-particle contributions to the total field at an observation point are transverse waves that can be characterized by the corresponding cumulative phases. It can then be argued that upon configurational averaging, the extreme sensitivity of the respective complex exponential phase factors on particle positions will zero out the contributions of all second-moment diagrams except those of the ladder diagrams (see, e.g., Section 8.11 of Ref. [4] or Section 18.2 of Ref. [9]). Strictly speaking, this argument cannot be made in the case of densely packed particles to which the far-field assumption can be inapplicable.

We thus have to conclude that as compared to the derivation given in Ref. [1], the present derivation is less self-consistent.

Finally we note that our primary objective has been to trace the first-principles origin of the radiative transfer theory. Therefore, we have not discussed numerical methods for solving the vector radiative transfer equation (or its simplified scalar version) and specific practical applications in various fields of science and engineering. This further information can be found in Refs. [4, 9, 13, 14] and numerous publications cited therein.

Appendix 1. Integral-operator notation and Fourier transforms of vector and dyadic functions

Considering the dyadic function $\overline{\mathbf{A}}(\mathbf{r},\mathbf{r}')$, the vector function $\mathbf{x}(\mathbf{r})$, and the dyadic function $\overline{\mathbf{X}}(\mathbf{r},\mathbf{r}')$, we define the vector function $(\overline{\mathbf{A}}\mathbf{x})(\mathbf{r})$ and the dyadic function $(\overline{\mathbf{A}}\,\overline{\mathbf{X}})(\mathbf{r},\mathbf{r}')$ through the linear transformations

$$(\overline{\mathbf{A}}\mathbf{x})(\mathbf{r}) \stackrel{\text{def}}{=} \int \overline{\mathbf{A}}(\mathbf{r},\mathbf{r}_1)\cdot\mathbf{x}(\mathbf{r}_1)\,\mathrm{d}^3\mathbf{r}_1, \tag{86}$$

and

$$(\overline{\mathbf{A}}\,\overline{\mathbf{X}})(\mathbf{r},\mathbf{r}') \stackrel{\text{def}}{=} \int \overline{\mathbf{A}}(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{X}}(\mathbf{r}_1,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}_1, \tag{87}$$

respectively, the entire three-dimensional space $\mathbb{R}^3$ serving as the integration domain. In general, we have

$$(\overline{\mathbf{A}}\,\overline{\mathbf{B}}\,\overline{\mathbf{X}})(\mathbf{r},\mathbf{r}') = \int \overline{\mathbf{A}}(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{B}}(\mathbf{r}_1,\mathbf{r}_2)\cdot\overline{\mathbf{X}}(\mathbf{r}_2,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}_2\mathrm{d}^3\mathbf{r}_1. \tag{88}$$

The linear operator $\overline{\mathbf{A}}$ acting on vectors and dyadics is an integral operator with the kernel $\overline{\mathbf{A}}(\mathbf{r},\mathbf{r}')$. Although we use the same notation, we assume that it can be inferred from the context if a quantity is an operator or a dyadic function.

The inverse dyadic function $\overline{\mathbf{A}}^{-1}(\mathbf{r},\mathbf{r}')$ of the dyadic function $\overline{\mathbf{A}}(\mathbf{r},\mathbf{r}')$ is defined by the relation

$$(\overline{\mathbf{A}}\,\overline{\mathbf{A}}^{-1})(\mathbf{r},\mathbf{r}') = (\overline{\mathbf{A}}^{-1}\overline{\mathbf{A}})(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\overline{\mathbf{I}}, \tag{89}$$

or explicitly,

$$\begin{aligned}\int \overline{\mathbf{A}}(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{A}}^{-1}(\mathbf{r}_1,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}_1 &= \int \overline{\mathbf{A}}^{-1}(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{A}}(\mathbf{r}_1,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}_1\\ &= \delta(\mathbf{r}-\mathbf{r}')\overline{\mathbf{I}},\end{aligned} \tag{90}$$

where $\delta(\mathbf{r}-\mathbf{r}')$ is the three-dimensional delta function and $\overline{\mathbf{I}}$ is the identity dyadic. As a result, we find

$$(\overline{\mathbf{A}}^{-1}\,\overline{\mathbf{A}}\,\overline{\mathbf{X}})(\mathbf{r},\mathbf{r}') = \int \delta(\mathbf{r}-\mathbf{r}_1)\overline{\mathbf{X}}(\mathbf{r}_1,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}_2 = \overline{\mathbf{X}}(\mathbf{r},\mathbf{r}'), \tag{91}$$

that is

$$\overline{\mathbf{A}}^{-1}\,\overline{\mathbf{A}}\,\overline{\mathbf{X}} = \overline{\mathbf{A}}\,\overline{\mathbf{A}}^{-1}\,\overline{\mathbf{X}} = \overline{\mathbf{X}}. \tag{92}$$

The Fourier transforms of the vector function $\mathbf{x}(\mathbf{r})$ and the dyadic function $\overline{\mathbf{X}}(\mathbf{r},\mathbf{r}')$ are defined respectively, by

$$\mathcal{F}(\mathbf{x})(\mathbf{p}) = \int \mathrm{e}^{-\mathrm{j}\mathbf{p}\cdot\mathbf{r}}\mathbf{x}(\mathbf{r})\mathrm{d}^3\mathbf{r}, \tag{93}$$

$$\mathcal{F}(\overline{\mathbf{X}})(\mathbf{p},\mathbf{p}') = \int \mathrm{e}^{-\mathrm{j}\mathbf{p}\cdot\mathbf{r}}\overline{\mathbf{X}}(\mathbf{r},\mathbf{r}')\mathrm{e}^{\mathrm{j}\mathbf{p}'\cdot\mathbf{r}'}\,\mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{r}, \tag{94}$$

and the inverse transformations are

$$\mathbf{x}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int \mathrm{e}^{\mathrm{j}\mathbf{p}\cdot\mathbf{r}} \mathcal{F}(\mathbf{x})(\mathbf{p}) \, \mathrm{d}^3\mathbf{p}, \tag{95}$$

$$\overline{\mathbf{X}}(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^6} \int \mathrm{e}^{\mathrm{j}\mathbf{p}\cdot\mathbf{r}} \mathcal{F}(\overline{\mathbf{X}})(\mathbf{p},\mathbf{p}') \mathrm{e}^{-\mathrm{j}\mathbf{p}'\cdot\mathbf{r}'} \, \mathrm{d}^3\mathbf{p}' \mathrm{d}^3\mathbf{p}. \tag{96}$$

The well-known representation for the Dirac delta function,

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3} \int \mathrm{e}^{\mathrm{j}\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} \, \mathrm{d}^3\mathbf{p}, \tag{97}$$

plays an important role in the calculation.

It is not hard to see that the Fourier transforms of $(\overline{\mathbf{A}}\mathbf{x})(\mathbf{r})$ and $(\overline{\mathbf{A}}\,\overline{\mathbf{X}})(\mathbf{r},\mathbf{r}')$ are, respectively,

$$\mathcal{F}(\overline{\mathbf{A}}\mathbf{x})(\mathbf{p}) = \frac{1}{(2\pi)^3} \int \mathcal{F}(\overline{\mathbf{A}})(\mathbf{p},\mathbf{p}') \cdot \mathcal{F}(\mathbf{x})(\mathbf{p}') \, \mathrm{d}^3\mathbf{p}', \tag{98}$$

$$\mathcal{F}(\overline{\mathbf{A}}\,\overline{\mathbf{X}})(\mathbf{p},\mathbf{p}') = \frac{1}{(2\pi)^3} \int \mathcal{F}(\overline{\mathbf{A}})(\mathbf{p},\mathbf{p}_1) \cdot \mathcal{F}(\overline{\mathbf{X}})(\mathbf{p}_1,\mathbf{p}') \, \mathrm{d}^3\mathbf{p}_1, \tag{99}$$

and that

$$\mathcal{F}(\overline{\mathbf{A}}\,\overline{\mathbf{A}}^{-1})(\mathbf{p},\mathbf{p}') = \mathcal{F}(\overline{\mathbf{A}}^{-1}\overline{\mathbf{A}})(\mathbf{p},\mathbf{p}') = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\bar{\mathbf{I}}. \tag{100}$$

Using the short-hand notation $\mathcal{F}(\mathbf{x})(\mathbf{p}) = \mathbf{x}_p(\mathbf{p})$ and $\mathcal{F}(\overline{\mathbf{X}})(\mathbf{p},\mathbf{p}') = \overline{\mathbf{X}}_p(\mathbf{p},\mathbf{p}')$, we summarize below the Fourier transforms of some special dyadic functions.

1. If $\overline{\mathbf{X}}(\mathbf{r},\mathbf{r}')$ is a *translation invariant* dyadic, i.e.,

   $$\overline{\mathbf{X}}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{X}}(\mathbf{r}-\mathbf{r}'), \tag{101}$$

   we have

   $$\overline{\mathbf{X}}_p(\mathbf{p},\mathbf{p}') = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\overline{\mathbf{X}}_p(\mathbf{p}), \tag{102}$$

   where

   $$\begin{aligned}\overline{\mathbf{X}}_p(\mathbf{p}) &= \int \overline{\mathbf{X}}(\mathbf{r}-\mathbf{r}')\mathrm{e}^{-\mathrm{j}\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}^3(\mathbf{r}-\mathbf{r}') \\ &= \int \overline{\mathbf{X}}(\mathbf{R})\mathrm{e}^{-\mathrm{j}\mathbf{p}\cdot\mathbf{R}} \, \mathrm{d}^3\mathbf{R}.\end{aligned} \tag{103}$$

   The Fourier transform of the vector function $\mathbf{a}(\mathbf{r}) = (\overline{\mathbf{X}}\mathbf{b})(\mathbf{r})$ is

   $$\mathbf{a}_p(\mathbf{p}) = \overline{\mathbf{X}}_p(\mathbf{p}) \cdot \mathbf{b}_p(\mathbf{p}), \tag{104}$$

   while the Fourier transform of the dyadic function $\overline{\mathbf{A}}(\mathbf{r},\mathbf{r}') = (\overline{\mathbf{X}}\,\overline{\mathbf{B}})(\mathbf{r},\mathbf{r}')$ is

   $$\overline{\mathbf{A}}_p(\mathbf{p},\mathbf{p}') = \overline{\mathbf{X}}_p(\mathbf{p}) \cdot \overline{\mathbf{B}}_p(\mathbf{p},\mathbf{p}'). \tag{105}$$

Taking the Fourier transform of Eq. (90), using Eq. (99) and (cf. Eqs. (94) and (97))

$$\mathcal{F}(\delta\overline{\mathbf{I}})(\mathbf{p},\mathbf{p}') = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\overline{\mathbf{I}}, \tag{106}$$

we obtain

$$\frac{1}{(2\pi)^3}\int \overline{\mathbf{X}}_p(\mathbf{p},\mathbf{p}_1)\cdot\overline{\mathbf{X}}_p^{-1}(\mathbf{p}_1,\mathbf{p}')\mathrm{d}^3\mathbf{p}_1 = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\overline{\mathbf{I}}, \tag{107}$$

whence, accounting for Eq. (102), we find

$$\overline{\mathbf{X}}_p(\mathbf{p})\cdot\overline{\mathbf{X}}_p^{-1}(\mathbf{p}) = \overline{\mathbf{I}}. \tag{108}$$

Thus, in the Fourier space, $\overline{\mathbf{X}}_p^{-1}$ is the inverse of $\overline{\mathbf{X}}_p$.

2. A dyadic function such that

$$\overline{\mathbf{X}}_i(\mathbf{r},\mathbf{r}') = \overline{\mathbf{X}}(\mathbf{r}-\mathbf{R}_i,\mathbf{r}'-\mathbf{R}_i) \tag{109}$$

is called *translational*. The Fourier transform of $\overline{\mathbf{X}}_i$ is

$$\overline{\mathbf{X}}_{ip}(\mathbf{p},\mathbf{p}') = \mathrm{e}^{-\mathrm{j}\mathbf{p}\cdot\mathbf{R}_i}\overline{\mathbf{X}}_p(\mathbf{p},\mathbf{p}')\mathrm{e}^{\mathrm{j}\mathbf{p}'\cdot\mathbf{R}_i}, \tag{110}$$

where $\overline{\mathbf{X}}_p(\mathbf{p},\mathbf{p}')$ is the Fourier transform of $\overline{\mathbf{X}}$. The Fourier transform of the vector $\mathbf{a}_i(\mathbf{r}) = (\overline{\mathbf{X}}_i\mathbf{b})(\mathbf{r})$ is

$$\mathbf{a}_{ip}(\mathbf{p}) = \frac{1}{(2\pi)^3}\int \mathrm{e}^{-\mathrm{j}(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}_i}\overline{\mathbf{X}}_p(\mathbf{p},\mathbf{p}_1)\cdot\mathbf{b}_p(\mathbf{p}_1)\,\mathrm{d}^3\mathbf{p}_1, \tag{111}$$

while the Fourier transform of the dyadic function $\overline{\mathbf{A}}_i(\mathbf{r},\mathbf{r}') = (\overline{\mathbf{X}}_i\,\overline{\mathbf{B}})(\mathbf{r},\mathbf{r}')$ is

$$\overline{\mathbf{A}}_{ip}(\mathbf{p},\mathbf{p}') = \frac{1}{(2\pi)^3}\int \mathrm{e}^{-\mathrm{j}(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}_i}\overline{\mathbf{X}}_p(\mathbf{p},\mathbf{p}_1)\cdot\overline{\mathbf{B}}_p(\mathbf{p}_1,\mathbf{p}')\,\mathrm{d}^3\mathbf{p}_1. \tag{112}$$

Appendix 2. Transition dyadic

Consider a homogeneous particle embedded in a lossless, homogeneous and isotropic medium. The wavenumber in the host medium is k_1, while the wavenumber inside the particle is $k_2 = \mathrm{m}k_1$, where m is the relative refractive index of the particle. The particle is centered at the origin of the coordinate system, and we denote by D_0 the domain occupied by the particle. The scattering potential of the particle $U_0(\mathbf{r}) = k_1^2(\mathrm{m}^2-1)\Theta(\mathbf{r})$, with

$$\Theta(\mathbf{r}) = \begin{cases} 1, & \mathbf{r}\in D_0 \\ 0, & \mathbf{r}\notin D_0 \end{cases},$$

can be elevated to a dyadic, the so-called potential dyadic, defined by

$$\overline{\mathbf{U}}_0(\mathbf{r},\mathbf{r}') = U_0(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\overline{\mathbf{I}},\ \ \mathbf{r},\mathbf{r}'\in\mathbb{R}^3. \tag{113}$$

In terms of $\overline{\mathbf{U}}_0$, the integral equations for the total field and the dyadic Green's function are

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \int \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{U}}_0(\mathbf{r}_1,\mathbf{r}_2) \cdot \mathbf{E}(\mathbf{r}_2)\,\mathrm{d}^3\mathbf{r}_2\mathrm{d}^3\mathbf{r}_1,\ \ \mathbf{r}\in\mathbb{R}^3, \tag{114}$$

and

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}') + \int \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{U}}_0(\mathbf{r}_1,\mathbf{r}_2) \cdot \overline{\mathbf{G}}(\mathbf{r}_2,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}_2\mathrm{d}^3\mathbf{r}_1,\ \ \mathbf{r},\mathbf{r}'\in\mathbb{R}^3, \tag{115}$$

respectively.

The transition dyadic $\overline{\mathbf{T}}$ is defined through the relation

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \int \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1) \cdot \overline{\mathbf{T}}(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathbf{E}_0(\mathbf{r}_2)\,\mathrm{d}^3\mathbf{r}_2\mathrm{d}^3\mathbf{r}_1,\ \ \mathbf{r}\in\mathbb{R}^3; \tag{116}$$

like for $\overline{\mathbf{U}}_0$, the support of the transition dyadic $\overline{\mathbf{T}}$ is D_0, that is,

$$\overline{\mathbf{T}}(\mathbf{r}_1,\mathbf{r}_2) = \overline{\mathbf{0}} \text{ for } \mathbf{r}_1 \text{ and } / \text{or } \mathbf{r}_2 \notin D_0.$$

The scattered field, given by

$$\mathbf{E}_{\mathtt{sct}}(\mathbf{r}) = \int \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{U}}_0(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathbf{E}(\mathbf{r}_2)\,\mathrm{d}^3\mathbf{r}_2\mathrm{d}^3\mathbf{r}_1,\ \ \mathbf{r}\in\mathbb{R}^3\setminus D_0, \tag{117}$$

and the internal field $\mathbf{E}_{\mathtt{int}}$ can be expressed in terms of the transition dyadic as

$$\mathbf{E}_{\mathtt{sct}}(\mathbf{r}) = \int \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{T}}(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathbf{E}_0(\mathbf{r}_2)\,\mathrm{d}^3\mathbf{r}_2\mathrm{d}^3\mathbf{r}_1,\ \ \mathbf{r}\in\mathbb{R}^3\setminus D_0, \tag{118}$$

and

$$\mathbf{E}_{\mathtt{int}}(\mathbf{r}) = \frac{1}{k_1^2(\mathrm{m}^2-1)}\int \overline{\mathbf{T}}(\mathbf{r},\mathbf{r}_1)\cdot\mathbf{E}_0(\mathbf{r}_1)\,\mathrm{d}^3\mathbf{r}_1,\ \ \mathbf{r}\in D_0, \tag{119}$$

respectively. Similarly, in terms of the transition dyadic, the integral equation for the dyadic Green's function is

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}') + \int \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{T}}(\mathbf{r}_1,\mathbf{r}_2)\cdot\overline{\mathbf{G}}_0(\mathbf{r}_2,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}_2\mathrm{d}^3\mathbf{r}_1. \tag{120}$$

The transition dyadic satisfies the Lippmann–Schwinger equation

$$\overline{\mathbf{T}}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{U}}_0(\mathbf{r},\mathbf{r}') + \int \overline{\mathbf{U}}_0(\mathbf{r},\mathbf{r}_1)\cdot\overline{\mathbf{G}}_0(\mathbf{r}_1,\mathbf{r}_2)\cdot\overline{\mathbf{T}}(\mathbf{r}_2,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}_2\mathrm{d}^3\mathbf{r}_1, \tag{121}$$

so that once $\overline{\mathbf{T}}$ is known, the scattered and internal fields can be computed by means of Eqs. (118) and (119), respectively.

Appendix 3. Relationship between the far-field scattering dyadic and the Fourier transform of the transition dyadic

As in Appendix 2, we consider the scattering by a homogeneous particle centered at the origin of the coordinate system. For an incident plane electromagnetic wave

$$\mathbf{E}_0(\mathbf{r}) = \boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}})\mathrm{e}^{\mathrm{j}\mathbf{k}_1\cdot\mathbf{r}}, \tag{122}$$

with $\mathbf{k}_1 = k_1\widehat{\mathbf{s}}$ and $\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) = \mathcal{E}_{0\theta}\widehat{\boldsymbol{\theta}}(\widehat{\mathbf{s}}) + \mathcal{E}_{0\varphi}\widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{s}})$, the scattered field is given by

$$\begin{aligned}\mathbf{E}_{\mathtt{sct}}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int \overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1)\\ \cdot\Big[\int \mathrm{e}^{\mathrm{j}\mathbf{p}_1\cdot\mathbf{r}_1}\overline{\mathbf{T}}_p(\mathbf{p}_1,\mathbf{k}_1)\cdot\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}})\,\mathrm{d}^3\mathbf{p}_1\Big]\,\mathrm{d}^3\mathbf{r}_1,\end{aligned} \tag{123}$$

where

$$\overline{\mathbf{T}}_p(\mathbf{p}_1,\mathbf{p}_2) = \int \mathrm{e}^{-\mathrm{j}\mathbf{p}_1\cdot\mathbf{r}_1}\overline{\mathbf{T}}(\mathbf{r}_1,\mathbf{r}_2)\mathrm{e}^{\mathrm{j}\mathbf{p}_2\cdot\mathbf{r}_2}\,\mathrm{d}^3\mathbf{r}_1\mathrm{d}^3\mathbf{r}_2, \tag{124}$$

is the Fourier transform of the transition dyadic. In the far-field region, the dyadic Green's function becomes

$$\overline{\mathbf{G}}_0(\mathbf{r},\mathbf{r}_1) = \big(\overline{\mathbf{I}} - \widehat{\mathbf{r}}\otimes\widehat{\mathbf{r}}\big)\frac{\mathrm{e}^{\mathrm{j}k_1 r}}{4\pi r}\mathrm{e}^{-\mathrm{j}k_1\widehat{\mathbf{r}}\cdot\mathbf{r}_1}, \tag{125}$$

in which case, Eq. (123) yields

$$\mathbf{E}_{\mathtt{sct}}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{j}k_1 r}}{4\pi r}\big(\overline{\mathbf{I}} - \widehat{\mathbf{r}}\otimes\widehat{\mathbf{r}}\big)\cdot\overline{\mathbf{T}}_p(k_1\widehat{\mathbf{r}},k_1\widehat{\mathbf{s}})\cdot\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}). \tag{126}$$

On the other hand, the scattered field can be expressed in terms of the far-field scattering dyadic

$$\overline{\mathbf{A}}(\widehat{\mathbf{r}},\widehat{\mathbf{s}}) = \sum_{\eta,\mu=\theta,\varphi}[\mathsf{S}(\widehat{\mathbf{r}},\widehat{\mathbf{s}})]_{\eta\mu}\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{r}})\otimes\widehat{\boldsymbol{\mu}}(\widehat{\mathbf{s}}), \tag{127}$$

where $\mathsf{S}(\widehat{\mathbf{r}},\widehat{\mathbf{s}})$ is the amplitude scattering matrix, according to

$$\mathbf{E}_{\mathtt{sct}}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{j}k_1 r}}{r}\overline{\mathbf{A}}(\widehat{\mathbf{r}},\widehat{\mathbf{s}})\cdot\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}),\ \ r\to\infty. \tag{128}$$

From Eqs. (126) and (128), the relation between the far-field scattering dyadic and the transition dyadic is

$$\overline{\mathbf{A}}(\widehat{\mathbf{r}},\widehat{\mathbf{s}})\cdot\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) = \frac{1}{4\pi}(\overline{\mathbf{I}} - \widehat{\mathbf{r}}\otimes\widehat{\mathbf{r}})\cdot\overline{\mathbf{T}}_p(k_1\widehat{\mathbf{r}},k_1\widehat{\mathbf{s}})\cdot\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}). \tag{129}$$

Putting successively $\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) = \mathcal{E}_{0\theta}\widehat{\boldsymbol{\theta}}(\widehat{\mathbf{s}})$ and $\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) = \mathcal{E}_{0\varphi}\widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{s}})$ in Eq. (129), we obtain

$$\overline{\mathbf{A}}(\widehat{\mathbf{r}},\widehat{\mathbf{s}}) = \frac{1}{4\pi}(\overline{\mathbf{I}} - \widehat{\mathbf{r}}\otimes\widehat{\mathbf{r}})\cdot\overline{\mathbf{T}}_p(k_1\widehat{\mathbf{r}},k_1\widehat{\mathbf{s}})\cdot(\overline{\mathbf{I}} - \widehat{\mathbf{s}}\otimes\widehat{\mathbf{s}}), \tag{130}$$

or equivalently,

$$\overline{\mathbf{A}}(\widehat{\mathbf{r}}, \widehat{\mathbf{s}}) = \frac{1}{4\pi} \overline{\mathbf{T}}_{p\mathtt{T}}(k_1 \widehat{\mathbf{r}}, k_1 \widehat{\mathbf{s}}), \tag{131}$$

where $\overline{\mathbf{T}}_{p\mathtt{T}}$ is the transverse component of the dyadic $\overline{\mathbf{T}}_p$ involving only the dyads $\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{r}}) \otimes \widehat{\boldsymbol{\mu}}(\widehat{\mathbf{s}})$, with $\eta, \mu = \theta, \varphi$. In matrix form, we use the representations (127) and

$$\overline{\mathbf{T}}_{p\mathtt{T}}(k_1 \widehat{\mathbf{r}}, k_1 \widehat{\mathbf{s}}) = \sum_{\eta,\mu=\theta,\varphi} [\mathsf{T}_{p\mathtt{T}}(k_1 \widehat{\mathbf{r}}, k_1 \widehat{\mathbf{s}})]_{\eta\mu} \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{r}}) \otimes \widehat{\boldsymbol{\mu}}(\widehat{\mathbf{s}}), \tag{132}$$

to obtain

$$\mathsf{S}(\widehat{\mathbf{r}}, \widehat{\mathbf{s}}) = \frac{1}{4\pi} \mathsf{T}_{p\mathtt{T}}(k_1 \widehat{\mathbf{r}}, k_1 \widehat{\mathbf{s}}). \tag{133}$$

Appendix 4. Characteristic waves method

Applying the operator $\nabla \times \nabla \times -k_1^2$ to the Dyson equation for the coherent field $\langle \mathbf{E} \rangle = \mathbf{E}_0 + \overline{\mathbf{G}}_0 \, \overline{\mathbf{M}} \, \langle \mathbf{E} \rangle$, taking the Fourier transform of the resulting equation, using the computation rule $\mathcal{F}(\nabla \times \mathbf{f})(\mathbf{p}) = \mathrm{j}\mathbf{p} \times \mathcal{F}(\mathbf{f})(\mathbf{p})$, where $\mathcal{F}(\mathbf{f})(\mathbf{p})$ is the Fourier transform of $\mathbf{f} = \mathbf{f}(\mathbf{r})$ at $\mathbf{p}$, assuming that for a statistical homogeneous medium, the dyadic mass operator is translation invariant, i.e., $\overline{\mathbf{M}}(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{M}}(\mathbf{r} - \mathbf{r}')$, which implies (see Appendix 1) $\mathcal{F}(\overline{\mathbf{M}})(\mathbf{p}, \mathbf{p}') = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \overline{\mathbf{M}}_p(\mathbf{p})$, we find [6]

$$[(p^2 - k_1^2)(\overline{\mathbf{I}} - \widehat{\mathbf{p}} \otimes \widehat{\mathbf{p}}) - k_1^2 \widehat{\mathbf{p}} \otimes \widehat{\mathbf{p}} - \overline{\mathbf{M}}_p(\mathbf{p})] \cdot \mathbf{E}_p(\mathbf{p}) = \mathbf{0}, \tag{134}$$

with $\mathcal{F}(\langle \mathbf{E} \rangle)(\mathbf{p}) = \mathbf{E}_p(\mathbf{p})$. This equation, which can be interpreted as an eigenvalue equation, shows that $\mathbf{E}_p$ is nonzero only if

$$\det[(p^2 - k_1^2)(\overline{\mathbf{I}} - \widehat{\mathbf{p}} \otimes \widehat{\mathbf{p}}) - k_1^2 \widehat{\mathbf{p}} \otimes \widehat{\mathbf{p}} - \overline{\mathbf{M}}_p(\mathbf{p})] = 0. \tag{135}$$

Here, the notation $\det(\overline{\mathbf{X}})$ should be understood as $\det(\mathsf{X})$, where X is the matrix associated with the dyadic $\overline{\mathbf{X}}$. Putting $\mathbf{p} = K\widehat{\mathbf{p}}$, the eigenvalues $K = K(\widehat{\mathbf{p}})$ satisfying the characteristic equation

$$\det[(K^2 - k_1^2)(\overline{\mathbf{I}} - \widehat{\mathbf{p}} \otimes \widehat{\mathbf{p}}) - k_1^2 \widehat{\mathbf{p}} \otimes \widehat{\mathbf{p}} - \overline{\mathbf{M}}_p(K\widehat{\mathbf{p}})] = 0, \tag{136}$$

are the values of the effective wavenumber (propagation constant) for the specified direction of propagation $\widehat{\mathbf{p}}$, while $\mathbf{E}_p(K\widehat{\mathbf{p}})$ satisfying Eq. (134) are the corresponding eigenvectors. In general, the dispersion equation (136) can have several solutions. In order to insure that the radiation condition at infinity is satisfied, only the solutions with a *positive imaginary part* are considered.

For the incidence propagation direction $\widehat{\mathbf{s}}$, let $K_n = K_n(\widehat{\mathbf{s}})$ be an eigenvalue solving the dispersion equation (136) with $\widehat{\mathbf{p}} = \widehat{\mathbf{s}}$, and let $\mathfrak{E}_n(\widehat{\mathbf{s}})$ be the corresponding eigenvector satisfying the eigenvalue equation (134) with $\mathbf{p} = K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}$ (that is, with $p = K_n(\widehat{\mathbf{s}})$ and $\widehat{\mathbf{p}} = \widehat{\mathbf{s}}$). Then,

$$\mathbf{E}_p(\mathbf{p}) = (2\pi)^3 \sum_n \delta[\mathbf{p} - K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}] \mathfrak{E}_n(\widehat{\mathbf{s}}) \tag{137}$$

satisfies Eq. (134), and consequently, the corresponding coherent field is

$$\langle \mathbf{E}(\mathbf{r}) \rangle = \sum_n \mathbf{E}_n(\mathbf{r}, \widehat{\mathbf{s}}), \tag{138}$$

$$\mathbf{E}_n(\mathbf{r}, \widehat{\mathbf{s}}) = \mathrm{e}^{\mathrm{j}K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}\cdot\mathbf{r}} \boldsymbol{\mathfrak{E}}_n(\widehat{\mathbf{s}}). \tag{139}$$

The wave $\mathbf{E}_n(\mathbf{r}, \widehat{\mathbf{s}})$ propagating in the direction $\widehat{\mathbf{s}}$ and satisfying the vector Helmholtz equation with wavenunber K_n is the characteristic wave, while the eigenvector $\boldsymbol{\mathfrak{E}}_n(\widehat{\mathbf{s}})$, which is *defined up to a multiplicative constant*, is the characteristic wave polarization associated with K_n.

For a discrete random medium with sparsely distributed particles, the approximation $K_n \approx k_1$, implying $\overline{\mathbf{M}}_p(K_n\widehat{\mathbf{s}}) \approx \overline{\mathbf{M}}_p(k_1\widehat{\mathbf{s}})$, can be assumed. Moreover, in this case, the far-field approximation, according to which the fields in the far zone are transverse, applies. Decomposing $\boldsymbol{\mathfrak{E}}_n(\widehat{\mathbf{s}})$ into a longitudinal and a transverse component $\boldsymbol{\mathfrak{E}}_{n\mathtt{L}}(\widehat{\mathbf{s}})$ and $\boldsymbol{\mathfrak{E}}_{n\mathtt{T}}(\widehat{\mathbf{s}})$, respectively, that is, $\boldsymbol{\mathfrak{E}}_n(\widehat{\mathbf{s}}) = \boldsymbol{\mathfrak{E}}_{n\mathtt{L}}(\widehat{\mathbf{s}}) + \boldsymbol{\mathfrak{E}}_{n\mathtt{T}}(\widehat{\mathbf{s}})$, with $\boldsymbol{\mathfrak{E}}_{n\mathtt{L}}(\widehat{\mathbf{s}}) = \mathfrak{E}_{ns}(\widehat{\mathbf{s}})\widehat{\mathbf{s}}$ and $\boldsymbol{\mathfrak{E}}_{n\mathtt{T}}(\widehat{\mathbf{s}}) = \mathfrak{E}_{n\theta}(\widehat{\mathbf{s}})\widehat{\boldsymbol{\theta}}(\widehat{\mathbf{s}}) + \mathfrak{E}_{n\varphi}(\widehat{\mathbf{s}})\widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{s}})$, the far-field approximation implies $\boldsymbol{\mathfrak{E}}_{n\mathtt{L}}(\widehat{\mathbf{s}}) = \mathbf{0}$. This results may also follow from the fact that in Eq. (134), k_1^2 is much larger than the $(\widehat{\mathbf{p}} \otimes \widehat{\mathbf{p}})$-, $(\widehat{\mathbf{p}} \otimes \widehat{\boldsymbol{\theta}}(\widehat{\mathbf{p}}))$-, and $(\widehat{\mathbf{p}} \otimes \widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{p}}))$-components of the dyadic $\overline{\mathbf{M}}_p(k_1\widehat{\mathbf{s}})$; in a matrix-form representation, the first equation in Eq. (134) yields $\mathfrak{E}_{ns}(\widehat{\mathbf{s}}) = 0$ [6]. Consequently, the matrix equation for the transverse part $\boldsymbol{\mathfrak{E}}_{n\mathtt{T}}(\widehat{\mathbf{s}})$ is

$$[(K_n^2 - k_1^2)\mathsf{I}_2 - \mathsf{M}_{p\mathtt{T}}(k_1\widehat{\mathbf{s}})] \begin{bmatrix} \mathfrak{E}_{n\theta}(\widehat{\mathbf{s}}) \\ \mathfrak{E}_{n\varphi}(\widehat{\mathbf{s}}) \end{bmatrix} = \mathsf{0}, \tag{140}$$

where I_2 is the two-dimensional identity matrix, $\mathsf{0}$ is the two-dimensional zero vector, and $\mathsf{M}_{p\mathtt{T}}(k_1\widehat{\mathbf{s}})$ is the matrix associated with the transverse component $\overline{\mathbf{M}}_{p\mathtt{T}}(\mathbf{p})$ of the dyadic $\overline{\mathbf{M}}_p(\mathbf{p})$. As shown in Ref. [6], for the direction of propagation $\widehat{\mathbf{s}}$, there are two effective wavenumbers $K_1(\widehat{\mathbf{s}})$ and $K_2(\widehat{\mathbf{s}})$, and accordingly, two transverse characteristic waves with polarizations $\boldsymbol{\mathfrak{E}}_{1\mathtt{T}}(\widehat{\mathbf{s}})$ and $\boldsymbol{\mathfrak{E}}_{1\mathtt{T}}(\widehat{\mathbf{s}})$. Furthermore, under the assumption $K_n \approx k_1$, we have $K_n^2 - k_1^2 \approx 2k_1(K_n - k_1)$, and the eigenvalue equation (140) yields

$$K_n \begin{bmatrix} \mathfrak{E}_{n\theta}(\widehat{\mathbf{s}}) \\ \mathfrak{E}_{n\varphi}(\widehat{\mathbf{s}}) \end{bmatrix} = \left[k_1\mathsf{I}_2 + \frac{1}{2k_1}\mathsf{M}_{p\mathtt{T}}(k_1\widehat{\mathbf{s}})\right] \begin{bmatrix} \mathfrak{E}_{n\theta}(\widehat{\mathbf{s}}) \\ \mathfrak{E}_{n\varphi}(\widehat{\mathbf{s}}) \end{bmatrix}. \tag{141}$$

Putting $\mathsf{A} = k_1\mathsf{I}_2 + (1/2k_1)\mathsf{M}_{p\mathtt{T}}(k_1\widehat{\mathbf{s}})$ and $\mathsf{x}_n = [\mathfrak{E}_{n\theta}(\widehat{\mathbf{s}})\, \mathfrak{E}_{n\varphi}(\widehat{\mathbf{s}})]^T$, so that $\mathsf{A}\mathsf{x}_n = K_n\mathsf{x}_n$, we see that $\mathsf{A}^2\mathsf{x}_n = K_n\mathsf{A}\mathsf{x}_n = K_n^2\mathsf{x}_n$, and that in general, $\mathsf{A}^m\mathsf{x}_n = K_n^m\mathsf{x}_n$ for any $m \geq 0$. This result implies $\exp(\mathrm{j}K_n s)\mathsf{x} = \exp(\mathrm{j}\mathsf{A}s)\mathsf{x}$, that is,

$$\mathrm{e}^{\mathrm{j}K_n s} \begin{bmatrix} \mathfrak{E}_{n\theta}(\widehat{\mathbf{s}}) \\ \mathfrak{E}_{n\varphi}(\widehat{\mathbf{s}}) \end{bmatrix} = \mathrm{e}^{\mathrm{j}[k_1\mathsf{I}_2 + \frac{1}{2k_1}\mathsf{M}_{p\mathtt{T}}(k_1\widehat{\mathbf{s}})]s} \begin{bmatrix} \mathfrak{E}_{n\theta}(\widehat{\mathbf{s}}) \\ \mathfrak{E}_{n\varphi}(\widehat{\mathbf{s}}) \end{bmatrix}, \tag{142}$$

with $s = s(\mathbf{r}, -\widehat{\mathbf{s}}) = \widehat{\mathbf{s}} \cdot (\mathbf{r} - \mathbf{r}_A)$. In dyadic representation, Eqs. (138), (139) and (142) imply

$$\begin{aligned} \langle \mathbf{E}(\mathbf{r}) \rangle &= \sum_{n=1}^{2} \mathrm{e}^{\mathrm{j}K_n(\widehat{\mathbf{s}})s(\mathbf{r},-\widehat{\mathbf{s}})} \mathrm{e}^{\mathrm{j}K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}\cdot\mathbf{r}_A} \boldsymbol{\mathfrak{E}}_{n\mathtt{T}}(\widehat{\mathbf{s}}) \\ &= \mathrm{e}^{\mathrm{j}[k_1\bar{\mathbf{I}} + \frac{1}{2k_1}\overline{\mathbf{M}}_{p\mathtt{T}}(k_1\widehat{\mathbf{s}})]s(\mathbf{r},-\widehat{\mathbf{s}})} \langle \mathbf{E}(\mathbf{r}_A) \rangle, \end{aligned} \tag{143}$$

where $\langle \mathbf{E}(\mathbf{r}_A) \rangle = \sum_{n=1}^{2} \mathrm{e}^{\mathrm{j}K_n(\widehat{\mathbf{s}})\widehat{\mathbf{s}}\cdot\mathbf{r}_A} \boldsymbol{\mathfrak{E}}_{n\mathtt{T}}(\widehat{\mathbf{s}})$. The coherent field given by Eq. (143) is a superposition of two plane electromagnetic waves propagating along the incidence direction but with different wavenumbers. Thus, in general, the coherent field is not a plane electromagnetic wave. That would be the case only if the effective wavenumber with the *smallest positive imaginary part* were considered (the characteristic wave with a larger imaginary parts is more strongly attenuated in the medium, and so, can be neglected).

From the Dyson equation, the dyadic mass operator is given by Eq. (13) in the coordinate space, and by

$$\overline{\mathbf{M}}_p(\mathbf{p}) = n_0 \overline{\mathbf{T}}_p(\mathbf{p}, \mathbf{p}) \tag{144}$$

in the Fourier space. In deriving Eq. (144), we used the fact that $\overline{\mathbf{M}}$ is translation invariant, and that $\overline{\mathbf{T}}_i$ is a translational dyadic, i.e., $\overline{\mathbf{T}}_i(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{T}}(\mathbf{r} - \mathbf{R}_i, \mathbf{r}' - \mathbf{R}_i)$. Taking into account the relationship between the far-field scattering dyadic and the Fourier transform of the transition dyadic given by Eq. (131), we obtain

$$\overline{\mathbf{M}}_{p\mathtt{T}}(k_1\widehat{\mathbf{s}}) = n_0 \overline{\mathbf{T}}_{p\mathtt{T}}(k_1\widehat{\mathbf{s}}, k_1\widehat{\mathbf{s}}) = 4\pi n_0 \overline{\mathbf{A}}(\widehat{\mathbf{s}}, \widehat{\mathbf{s}}). \tag{145}$$

Finally, substitution of Eq. (145) in Eq. (143) gives

$$\langle \mathbf{E}(\mathbf{r}) \rangle = \exp\Big\{ \mathrm{j} \Big[k_1 \overline{\mathbf{I}} + \frac{2\pi}{k_1} n_0 \overline{\mathbf{A}}(\widehat{\mathbf{s}}, \widehat{\mathbf{s}}) \Big] s(\mathbf{r}, -\widehat{\mathbf{s}}) \Big\} \cdot \langle \mathbf{E}(\mathbf{r}_A) \rangle, \tag{146}$$

which is Eq. (46). In summary, the representation (146) has been obtained under the assumptions and approximations which are representative of a discrete random medium with a sparse concentration of particles: the far-field approximation, the assumption that the positions of the particles are statistically independent, and the Twersky approximation (the last two assumptions essentially, yield Eq. (144)). In addition, we supposed that the coherent field propagates along the incidence direction, and that the effective wavenumber is close to that of the background medium.

Appendix 5. Solution of the Foldy integral equation for the coherent field

To solve the Foldy integral equation for the coherent field,

$$\mathbf{E}_{\mathsf{c}}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0 \int_D g_0(r_i) \overline{\mathbf{A}}(\widehat{\mathbf{r}}_i, \widehat{\mathbf{s}}) \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{R}_i) \, \mathrm{d}^3\mathbf{R}_i, \tag{147}$$

we assume a particular form for $\mathbf{E}_{\mathsf{c}}(\mathbf{r})$. Taking into account that the incident field can be expressed as $\mathbf{E}_0(\mathbf{r}) = \exp[\mathrm{j}k_1 s(\mathbf{r})]\mathbf{E}_0(\mathbf{r}_A)$, where $s(\mathbf{r})$ stands for $s(\mathbf{r}, -\widehat{\mathbf{s}})$ hereinafter, we suppose that $\mathbf{E}_{\mathsf{c}}(\mathbf{r})$ has a similar dependency on $\mathbf{E}_{\mathsf{c}}(\mathbf{r}_A)$. More specifically, we set

$$\mathbf{E}_{\mathsf{c}}(\mathbf{r}) = \exp\big[\mathrm{j}\overline{\mathbf{X}}(\widehat{\mathbf{s}}) s(\mathbf{r})\big] \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}_A) \tag{148}$$

for some unknown dyadic $\overline{\mathbf{X}}$ depending on the propagation direction of the incident wave $\widehat{\mathbf{s}}$. Assuming $\mathbf{E}_{\mathsf{c}}(\mathbf{r}_A) = \mathbf{E}_0(\mathbf{r}_A)$, and representing $\overline{\mathbf{X}}$ as

$$\overline{\mathbf{X}}(\widehat{\mathbf{s}}) = k_1 \bar{\mathbf{I}} + \frac{2\pi}{k_1} n_0 \overline{\mathbf{W}}(\widehat{\mathbf{s}}), \tag{149}$$

gives

$$\mathbf{E}_{\mathsf{c}}(\mathbf{r}) = \exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{W}}(\widehat{\mathbf{s}}) s(\mathbf{r})\Big] \cdot \mathbf{E}_0\left(\mathbf{r}\right), \tag{150}$$

where $\overline{\mathbf{W}}(\widehat{\mathbf{s}})$ is now the unknown dyadic to be determined. Setting $\mathbf{r} = \mathbf{R}_i$ in Eq. (150) yields

$$\mathbf{E}_{\mathsf{c}}(\mathbf{R}_i) = \exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{W}}(\widehat{\mathbf{s}}) s(\mathbf{R}_i)\Big] \cdot \mathbf{E}_0(\mathbf{R}_i). \tag{151}$$

Substituting Eq. (151) in Eq. (147), making the change of variable $\mathbf{R}_i = \mathbf{r} + \mathbf{p}$, which implies $\mathbf{p} = -\mathbf{r}_i$ and $\mathrm{d}^3\mathbf{R}_i = \mathrm{d}^3\mathbf{p}$, using the partial results

$$\mathbf{E}_0(\mathbf{R}_i) = \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{s}}\cdot\mathbf{p}} \mathbf{E}_0\left(\mathbf{r}\right) \tag{152}$$

and

$$\begin{aligned} &\exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{W}}(\widehat{\mathbf{s}}) s(\mathbf{R}_i)\Big] \\ &= \exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{W}}(\widehat{\mathbf{s}})(\mathbf{p}\cdot\widehat{\mathbf{s}})\Big] \cdot \exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{W}}(\widehat{\mathbf{s}}) s(\mathbf{r})\Big], \end{aligned} \tag{153}$$

and finally, accounting of Eq. (150), we find

$$\begin{aligned} \mathbf{E}_{\mathsf{c}}(\mathbf{r}) = \mathbf{E}_0\left(\mathbf{r}\right) + n_0 \Big\{ \int_D & \frac{\mathrm{e}^{\mathrm{j}k_1 p}}{p} \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{s}}\cdot\mathbf{p}} \overline{\mathbf{A}}(-\widehat{\mathbf{p}}, \widehat{\mathbf{s}}) \\ & \cdot \exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{W}}(\widehat{\mathbf{s}})(\mathbf{p}\cdot\widehat{\mathbf{s}})\Big] \,\mathrm{d}^3\mathbf{p} \Big\} \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}). \end{aligned} \tag{154}$$

The asymptotic expansion of a plane wave in spherical waves [12]

$$\mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{s}}\cdot\mathbf{p}} = \mathrm{j}\frac{2\pi}{k_1 p} \Big[\delta(\widehat{\mathbf{s}} + \widehat{\mathbf{p}})\mathrm{e}^{-\mathrm{j}k_1 p} - \delta(\widehat{\mathbf{s}} - \widehat{\mathbf{p}})\mathrm{e}^{\mathrm{j}k_1 p}\Big], \tag{155}$$

where $\delta(\widehat{\mathbf{s}} - \widehat{\mathbf{p}})$ is the solid-angle delta function, then yields

$$\begin{aligned} \mathbf{E}_{\mathsf{c}}(\mathbf{r}) = \mathbf{E}_0\left(\mathbf{r}\right) &+ \mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{A}}(\widehat{\mathbf{s}}, \widehat{\mathbf{s}}) \\ & \cdot \Big\{ \int_0^{s(\mathbf{r})} \exp\Big[-\mathrm{j}\frac{2\pi}{k_1} n_0 \overline{\mathbf{W}}(\widehat{\mathbf{s}}) p\Big] \,\mathrm{d}p \Big\} \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}). \end{aligned} \tag{156}$$

Left-multiplying the above equation by $\widehat{\mathbf{s}}$, using $\widehat{\mathbf{s}} \cdot \mathbf{E}_0\left(\mathbf{r}\right) = 0$ and $\widehat{\mathbf{s}} \cdot \overline{\mathbf{A}}(\widehat{\mathbf{s}}, \widehat{\mathbf{s}}) = \mathbf{0}$, we obtain $\widehat{\mathbf{s}} \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}) = 0$. Left-multiplying now Eq. (150) by $\widehat{\mathbf{s}}$, using $\widehat{\mathbf{s}} \cdot \mathbf{E}_{\mathsf{c}}(\mathbf{r}) = 0$, and employing a series representation of the dyadic exponential, we find

$\widehat{\mathbf{s}}\cdot\overline{\mathbf{W}}(\widehat{\mathbf{s}}) = \mathbf{0}$. We further make the assumption that $\overline{\mathbf{W}}(\widehat{\mathbf{s}})\cdot\widehat{\mathbf{s}} = \mathbf{0}$, or equivalently, that $\overline{\mathbf{W}}(\widehat{\mathbf{s}})$ is a transverse dyadic, i.e.,

$$\overline{\mathbf{W}}(\widehat{\mathbf{s}}) = \sum_{\eta,\mu=\theta,\varphi} [\mathsf{W}(\widehat{\mathbf{s}})]_{\eta\mu} \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}) \otimes \widehat{\boldsymbol{\mu}}(\widehat{\mathbf{s}}). \tag{157}$$

Because the coherent field is a transverse field, we write

$$\mathbf{E}_{\mathsf{c}}(\mathbf{r}) = E_{\mathsf{c}\theta}(\mathbf{r})\hat{\boldsymbol{\theta}}(\widehat{\mathbf{s}}) + E_{\mathsf{c}\varphi}(\mathbf{r})\hat{\boldsymbol{\varphi}}(\widehat{\mathbf{s}}), \tag{158}$$

and define the two-element column vector $\mathsf{E}_{\mathsf{c}}(\mathbf{r})$ according to $\mathsf{E}_{\mathsf{c}}(\mathbf{r}) = [E_{\mathsf{c}\theta}(\mathbf{r})\, E_{\mathsf{c}\varphi}(\mathbf{r})]^T$. As a result, the matrix equation associated with the dyadic equation (156) is

$$\begin{aligned} \mathsf{E}_{\mathsf{c}}(\mathbf{r}) = {} & \mathsf{E}_0(\mathbf{r}) + \mathrm{j}\frac{2\pi}{k_1} n_0 \mathsf{A}(\widehat{\mathbf{s}},\widehat{\mathbf{s}}) \\ & \times \Big\{ \int_0^{s(\mathbf{r})} \exp\Big[-\mathrm{j}\frac{2\pi}{k_1} n_0 \mathsf{W}(\widehat{\mathbf{s}}) p\Big]\, \mathrm{d}p \Big\} \mathsf{E}_{\mathsf{c}}(\mathbf{r}), \end{aligned} \tag{159}$$

while the matrix equation associated with the dyadic equation (150) is

$$\mathsf{E}_{\mathsf{c}}(\mathbf{r}) = \exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \mathsf{W}(\widehat{\mathbf{s}}) s(\mathbf{r})\Big] \mathsf{E}_0(\mathbf{r}). \tag{160}$$

The integral in Eq. (159) is

$$\begin{aligned} & \int_0^{s(\mathbf{r})} \exp\Big[-\mathrm{j}\frac{2\pi}{k_1} n_0 \mathsf{W}(\widehat{\mathbf{s}}) p\Big]\, \mathrm{d}p \\ & = \Big(\mathrm{j}\frac{2\pi}{k_1} n_0\Big)^{-1} \mathsf{W}^{-1}(\widehat{\mathbf{s}}) \Big\{ \mathsf{I}_2 - \exp\Big[-\mathrm{j}\frac{2\pi}{k_1} n_0 \mathsf{W}(\widehat{\mathbf{s}}) s(\mathbf{r})\Big] \Big\}, \end{aligned} \tag{161}$$

and we obtain

$$\begin{aligned} \mathsf{E}_{\mathsf{c}}(\mathbf{r}) = {} & \mathsf{E}_0(\mathbf{r}) + \mathsf{A}(\widehat{\mathbf{s}},\widehat{\mathbf{s}}) \mathsf{W}^{-1}(\widehat{\mathbf{s}}) \\ & \times \Big\{ \mathsf{I}_2 - \exp\Big[-\mathrm{j}\frac{2\pi}{k_1} n_0 \mathsf{W}(\widehat{\mathbf{s}}) s(\mathbf{r})\Big] \Big\} \mathsf{E}_{\mathsf{c}}(\mathbf{r}). \end{aligned} \tag{162}$$

Finally, by means of Eq. (160), we get

$$\begin{aligned} 0 = {} & [\mathsf{I}_2 - \mathsf{A}(\widehat{\mathbf{s}},\widehat{\mathbf{s}}) \mathsf{W}^{-1}(\widehat{\mathbf{s}})] \\ & \times \Big\{ \exp\Big[\mathrm{j}\frac{2\pi}{k_1} n_0 \mathsf{W}(\widehat{\mathbf{s}}) s(\mathbf{r})\Big] - \mathsf{I}_2 \Big\} \mathsf{E}_0(\mathbf{r}), \end{aligned} \tag{163}$$

and, since $\mathsf{E}_0(\mathbf{r})$ is arbitrary, the solution of Eq. (163) is $\mathsf{W}(\widehat{\mathbf{s}}) = \mathsf{A}(\widehat{\mathbf{s}},\widehat{\mathbf{s}})$. Applying this result to Eqs. (148) and (149), we find that the coherent field is as in Eq. (46).

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